Math 462 Final review questions

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April 24, 2015

(1) (a) Find a Mobius transformation

$$f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad f(z) = \frac{az+b}{cz+d}$$

such that $f(2) = 0$, $f(4) = 1$, and $f(1+i) = \infty$.
[Here $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.]

(b) Let C be the circle in the complex plane $\mathbb{C}$ with center $1+i$ and radius 1. Find a Mobius transformation

$$g\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

such that $g(C) = \mathbb{R} \cup \{\infty\}$ (the x-axis).

(2) Let $S^2 \subset \mathbb{R}^3$ be the sphere with center the origin and radius 1, and

$$\bar{F}\colon S^2 \to \mathbb{R}^2 \cup \{\infty\}$$

the map given by stereographic projection from the north pole $N = (0,0,1)$ to the xy-plane. Compute the image $\bar{F}(C)$ of the circle $C \subset S^2$ given by $C = \Pi \cap S^2$ where $\Pi \subset \mathbb{R}^3$ is the plane given by the following equations.

(a) $x + y + 2z = 2$.

(b) $4x + 2y + 5z = 6$.

[Hint: Recall the formulas

$$F(x,y,z) = \frac{1}{1-z}(x,y)$$

and

$$F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1).$$

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(3) Consider stereographic projection $\bar{F}\colon S^2 \to \mathbb{R}^2 \cup \{\infty\}$ as above. If we define a notion of distance on the extended plane $\mathbb{R}^2 \cup \{\infty\}$ corresponding to the spherical distance on S^2 under stereographic projection, then the length of a curve

$$\gamma\colon [a,b] \to \mathbb{R}^2, \quad \gamma(t) = (x(t), y(t))$$

is given by the formula

$$\text{length}(\gamma) = \int_a^b \frac{2\sqrt{x'(t)^2 + y'(t)^2}}{1 + x(t)^2 + y(t)^2} dt.$$

Check this formula by using it to compute the length of the following curves:

(a) The shortest path on S^2 from the south pole $(0,0,-1)$ to the point $(1,0,0)$ (given by the shorter arc of the great circle through the two points).

(b) The spherical circle $C = S^2 \cap \Pi$ where $\Pi \subset \mathbb{R}^3$ is the plane with equation $z = 3/5$.

(4) Let $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about the x-axis through angle $\pi/2$ counterclockwise as viewed from the positive x-axis. Let $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about the z-axis through angle $\pi/2$ counterclockwise as viewed from the positive z-axis. Using quaternions or otherwise, give a geometric description of the composition $T_2 \circ T_1$.

(5) Let $\mathcal{H} = \{x + iy \mid y > 0\}$ denote the upper half plane model of the hyperbolic plane.

(a) Compute the hyperbolic line $L \subset \mathcal{H}$ passing through the points $1 + i$ and $5 + 3i$.

(b) Compute the hyperbolic line $M \subset \mathcal{H}$ through the point $2i$ in the direction $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(6) (a) Determine a hyperbolic isometry $f\colon \mathcal{H} \to \mathcal{H}$ such that $f(2+3i) = 1+4i$.

(b) Let $L \subset \mathcal{H}$ be the hyperbolic line given by the circle with center 3 and radius 4, and $M \subset \mathcal{H}$ the hyperbolic line given by the y-axis. Determine a hyperbolic isometry $g\colon \mathcal{H} \to \mathcal{H}$ such that $g(L) = M$.

(7) Let $z_1 = -4+3i$ and $z_2 = 3+4i$.

(a) Determine the hyperbolic line L passing through z_1 and z_2.

(b) Compute the hyperbolic distance $d_{\mathcal{H}}(z_1, z_2)$ (the hyperbolic length of the shortest path connecting z_1 and z_2 given by L).

(c) Compute the hyperbolic length of the Euclidean line segment in $\mathcal{H}$ connecting z_1 and z_2, and check that it is greater than the length of the hyperbolic line segment computed in (b).

[Hint: (b) Write down a hyperbolic isometry f such that f sends L to the y-axis. Now use the formula $d_{\mathcal{H}}(iy_1, iy_2) = \ln(y_2/y_1)$ for $y_1 < y_2$. (c) Recall that the hyperbolic length of a path

$$\gamma\colon [a,b] \to \mathcal{H}, \quad \gamma(t) = x(t) + iy(t)$$

is given by

$$\text{length}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt.$$

Also note that the line segment from z_1 to z_2 can be parametrized by setting

$$\gamma\colon [0,1] \to \mathcal{H}, \quad \gamma(t) = z_1 + t(z_2 - z_1).$$

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(8) Let $T \subset \mathcal{H}$ be a hyperbolic triangle.

(a) Show that the area of T is less than π.

(b) Now suppose two of the sides of T are given by the y-axis and the circle with center the origin and radius 1. Show that the area of T is less than $\pi/2$.

(9) Find a formula for the hyperbolic reflection $f\colon \mathcal{H} \to \mathcal{H}$ in the hyperbolic line L given by the circle with center 3 and radius 2.

[Hint: First find a hyperbolic isometry $h\colon \mathcal{H} \to \mathcal{H}$ such that $h(L)$ is the hyperbolic line given by the circle C with center the origin and radius 1. (Here we can take $h(z) = az + b$ for some $a, b \in \mathbb{R}$, $a > 0$.) Now $f = h^{-1} \circ g \circ h$ where $g\colon \mathcal{H} \to \mathcal{H}$ is the hyperbolic reflection in $h(L)$ (why?), that is, g is given by inversion in the circle C. Finally use the known formula $g(z) = 1/\bar{z} = z/|z|^2$ to obtain a formula for f.]

(10) In each of the following cases, determine the fixed points of the hyperbolic isometry $f\colon \mathcal{H} \to \mathcal{H}$ in $\mathcal{H}$ and $\partial\mathcal{H}$. (Here $\partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$ denotes the boundary of $\mathcal{H}$ in $\mathbb{C} \cup \{\infty\}$.) Deduce the type of the isometry f in the classification of hyperbolic isometries.

(a) $f(z) = \frac{1+z}{1-z}$.
(b) $f(z) = \frac{z}{2z-1}$.
(c) $f(z) = \frac{5z-18}{2z-7}$.

(11) Consider the hyperbolic isometry

$$T\colon \mathcal{H} \to \mathcal{H}, \quad T(z) = \frac{z-4}{z+5}$$

(a) Give a geometric description of T and draw a diagram illustrating the effect of T.
(b) Find an expression $T = f^{-1} \circ S \circ f$ where $f\colon \mathcal{H} \to \mathcal{H}$ is a hyperbolic isometry and $S\colon \mathcal{H} \to \mathcal{H}$ is a hyperbolic isometry given by *either* $S(z) = az$ for some $a \in \mathbb{R}$ with $a > 0$, *or* $S(z) = z + b$ for some $b \in \mathbb{R}$.

(12) Consider the hyperbolic isometry

$$T\colon \mathcal{H} \to \mathcal{H}, \quad T(z) = \frac{z-8}{z-5}.$$

(a) Give a geometric description of T and draw a diagram illustrating the effect of T.
(b) Find an expression $T = f^{-1} \circ S \circ f$ where $f\colon \mathcal{H} \to \mathcal{H}$ is a hyperbolic isometry and $S\colon \mathcal{H} \to \mathcal{H}$ is a hyperbolic isometry given by *either* $S(z) = az$ for some $a \in \mathbb{R}$ with $a > 0$, *or* $S(z) = z + b$ for some $b \in \mathbb{R}$.