Solutions to 462 Midterm, 3/11/15.

Q1 a) $R = \|\overrightarrow{OP}\| = \left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}.$

b) $C :=$ great circle passing thru P & Q.

Then $C = \Pi \cap S^2$, where Π is the plane thru $O, P, \& Q$.

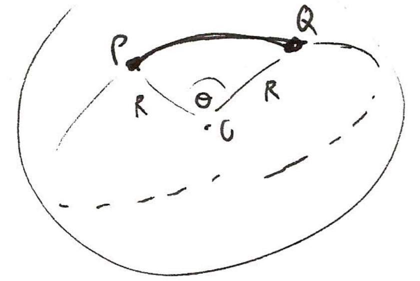
Π is given by equation $\underline{x} \cdot \underline{n} = 0$, where $\underline{n}$ is a normal vector for Π.

We can take $\underline{n} = \overrightarrow{OP} \times \overrightarrow{OQ} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$

So Π has equation $-x+y+z = 0.$

c) The shortest path from P to Q on the sphere S^2 is given by the (shorter) arc of the great circle thru P & Q. Its length is

$$d_{S^2}(P,Q) = R \cdot \theta$$



where θ is the angle between $\overrightarrow{OP}$ & $\overrightarrow{OQ}$.

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = \|\overrightarrow{OP}\| \cdot \|\overrightarrow{OQ}\| \cdot \cos(\theta) = R^2 \cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\overrightarrow{OP} \cdot \overrightarrow{OQ}}{R^2} \right) = \cos^{-1}\left(\frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{2}^2} \right)$$

$$= \cos^{-1}\left(\tfrac{1}{2}\right) = \pi/3.$$

$$\Rightarrow d_{S^2}(P,Q) = \sqrt{2} \cdot \pi/3.$$

Q2 (a) As in Q1(b) we can compute the equations of the sides of the spherical triangle ABC using the cross product:-

$$\vec{OA} \times \vec{OB} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

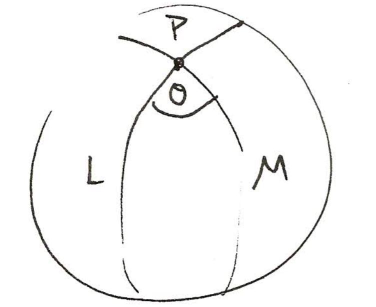
$\Rightarrow$ equation of plane Π_{AB} containing O, A & B is $-y+z=0$.

(note we can ignore the factor $1/\sqrt{2}$ here – this would just scale the equation).

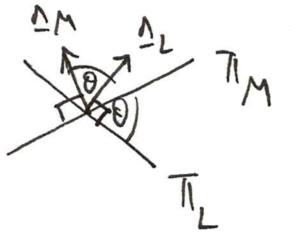
Similarly for BC & AC.

Or we can just observe directly that Π_{BC} has equation $x=0$ & Π_{AC} has equation $y=0$.

(b)



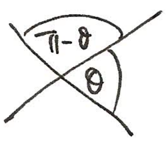
When two spherical lines meet at a point P, the angle between them is the same as the (dihedral) angle between the corresponding planes Π_L, Π_M, which is the same as the angle between their normal vectors



We can compute θ using the dot product:

$$\theta = \cos^{-1}\left(\underline{n}_L \cdot \underline{n}_M \Big/ \|\underline{n}_L\| \cdot \|\underline{n}_M\| \right).$$

[Note there is an ambiguity in the angle here:- θ vs. $\pi - \theta$



We are told here that the angle we want is $\leq \pi/2$, so we should take the smaller angle (equivalently $\cos\theta \geq 0$)]

Now compute:

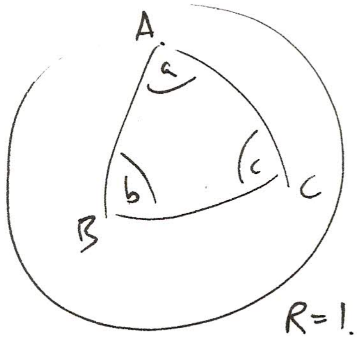
$$a = \cos^{-1}\left(\frac{\underline{n}_{AB}\cdot\underline{n}_{AC}}{\|\underline{n}_{AB}\|\cdot\|\underline{n}_{AC}\|}\right) \overset{\text{from (a)}}{=} \cos^{-1}\left(\frac{\begin{pmatrix}0\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{\sqrt{2}\cdot 1}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = 3\pi/4.$$

$\rightsquigarrow$ actually $a = \pi/4$ (see remark on p.2).

$$b = \cos^{-1}\left(\frac{\underline{n}_{AB}\cdot\underline{n}_{BC}}{\|\underline{n}_{AB}\|\cdot\|\underline{n}_{BC}\|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}0\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{2}\cdot 1}\right) = \cos^{-1}(0) = \pi/2.$$

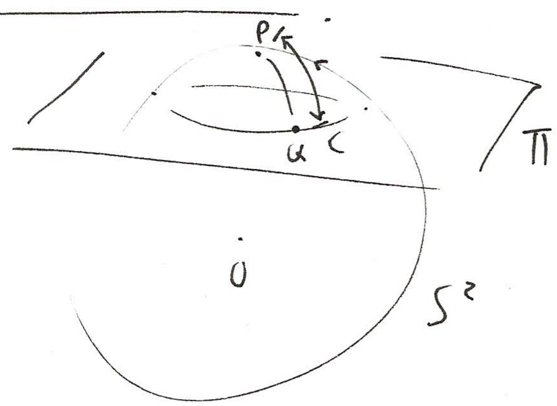
$$c = \cos^{-1}\left(\frac{\underline{n}_{AC}\cdot\underline{n}_{BC}}{\|\underline{n}_{AC}\|\cdot\|\underline{n}_{BC}\|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{1\cdot 1}\right) = \cos^{-1}(0) = \pi/2.$$

c) $a+b+c = \pi + \text{Area}(\Delta ABC)$ (proved in class)



$$\Rightarrow \quad \text{Area}(\Delta ABC) = a+b+c-\pi$$
$$= \tfrac{\pi}{4} + \tfrac{\pi}{2} + \tfrac{\pi}{2} - \pi = \tfrac{\pi}{4}.$$

Q3 (a)

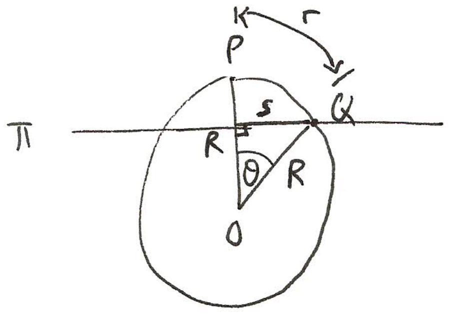


Spherical circle $C \subset S^2$ is $C = \Pi \cap S^2$ for $\Pi \subset \mathbb{R}^3$ some plane, not necessarily thru O.

Equivalently, C is the set of points $\alpha \in S^2$ at fixed spherical distance r from a given point $P \in S^2$.

Here r is the spherical <u>radius</u> of C and P is its spherical center.

Drawing a slice thru O, P, Q:



$r = R\cdot\theta \quad \left(= 2\pi R \cdot \theta/2\pi\right)$

$s = R\sin\theta$ = radius of C regarded as a circle in the plane $\mathbb{R}^2$.

Now circumference of $C = 2\pi\cdot s = 2\pi R\sin\theta = \boxed{2\pi R\sin(r/R)}$

(b) $\sin x \approx x - x^3/6$ for x small.

(this comes from the power series expansion for $\sin x$:-

$$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots = x - x^3/6 + \dots\ )$$

So, if r is small in comparison to R, r/R is small

and the circumference of C $A = 2\pi R\sin(r/R) \approx 2\pi R\left(r/R - \frac{1}{6}\cdot(r/R)^3\right)$

Dividing by the circumference $B = 2\pi r$ of a circle in the plane of radius r

gives $$\frac{A}{B} \approx \frac{2\pi R\left(r/R - \frac{1}{6}(r/R)^3\right)}{2\pi r} = R/r\left(r/R - \frac{1}{6}(r/R)^3\right) = \boxed{1 - \frac{1}{6}(r/R)^2}$$

This is a good approximation for r/R small.

(A weaker approximation is obtained using $\sin x \approx x$ for x small, that gives $A/B \approx 1$. This corresponds to the fact that the sphere S^2 appears flat at small scales).

Q4. (a)

Let $U: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be rotation about the origin thru angle $\theta = \pi$ counterclockwise.

Then $U(\underline{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \underline{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \underline{x} \; (= -\underline{x}).$

Now $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T(\underline{x}) = U\left(\underline{x} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\left(\underline{x} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

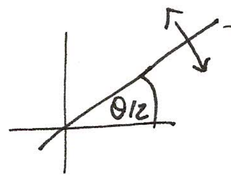
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\underline{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\underline{x} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}}.$$

(b). Let $U: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$. (i.e. the line parallel to the given line $y = x + 3$ passing thru the origin).

Then $U(\underline{x}) = A \cdot \underline{x}$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(to compute A: $U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $U\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. these give the columns of A. (235))

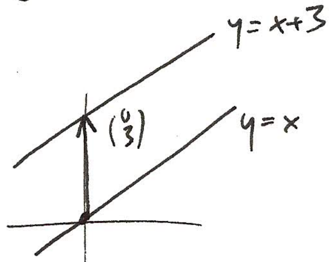
OR: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, where 

line of reflection.

For us, $\theta/2 = \pi/4$, $\theta = \pi/2$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. ✓

Now let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x+3$.

Then $V(\underline{x}) = U\left(\underline{x} - \begin{pmatrix}0\\3\end{pmatrix}\right) + \begin{pmatrix}0\\3\end{pmatrix}$



$= \begin{pmatrix}0&1\\1&0\end{pmatrix}\left(\underline{x} - \begin{pmatrix}0\\3\end{pmatrix}\right) + \begin{pmatrix}0\\3\end{pmatrix}$

$= \begin{pmatrix}0&1\\1&0\end{pmatrix}\underline{x} - \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}0\\3\end{pmatrix}$

$= \begin{pmatrix}0&1\\1&0\end{pmatrix}\underline{x} + \begin{pmatrix}-3\\3\end{pmatrix}$

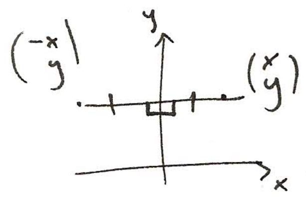
Finally $T(\underline{x}) = V(\underline{x}) + \underbrace{5\cdot\begin{pmatrix}1\\1\end{pmatrix}} = \boxed{\begin{pmatrix}0&1\\1&0\end{pmatrix}\underline{x} + \begin{pmatrix}2\\8\end{pmatrix}.}$

vector in direction of the line, scaled so that length $5\sqrt{2}$ and in the direction of increasing x

Q5 (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4-x\\y+5\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}4\\5\end{pmatrix}.$

Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad U\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}. = \begin{pmatrix}-x\\y\end{pmatrix}$

Observe that U is reflection in the y-axis :-



(OR: Compute $\det\begin{pmatrix}-1&0\\0&1\end{pmatrix} = -1 \Rightarrow$ reflection

$\begin{pmatrix}-1&0\\0&1\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \Rightarrow \theta = \pi \Rightarrow \theta/2 = \pi/2$

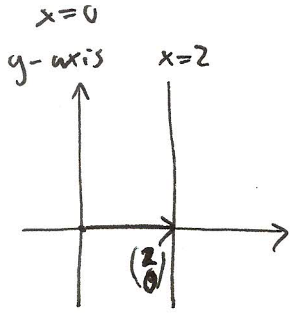
$\Rightarrow$ line of reflection is y-axis (see p. 5, bottom.))

Now, decompose translation $\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix}$

normal to line of reflection $\qquad$ parallel to line of reflection

Then $T(\underline{x}) = U(\underline{x}) + \begin{pmatrix}4\\5\end{pmatrix}$

$= \left(U(\underline{x}) + \begin{pmatrix}4\\0\end{pmatrix}\right) + \begin{pmatrix}0\\5\end{pmatrix}$



$$= \underbrace{\left(U\left(\underline{x} - \begin{pmatrix}2\\0\end{pmatrix}\right) + \begin{pmatrix}2\\0\end{pmatrix}\right)}_{\text{reflection in } x=2} + \begin{pmatrix}0\\5\end{pmatrix}$$

$\begin{pmatrix}2\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}4\\0\end{pmatrix}$

$U\begin{pmatrix}2\\0\end{pmatrix} = -\begin{pmatrix}2\\0\end{pmatrix}$

translation, parallel to line of reflection, thru distance 5 in direction of increasing y.

i.e. T is a <u>glide reflection</u>.

b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3-y\\x+7\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}3\\7\end{pmatrix}$

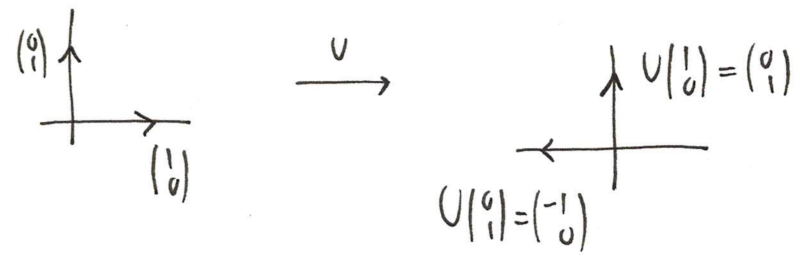
Let $U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $U\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$\det\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = +1 \Rightarrow U$ is a rotation, $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

$\Rightarrow \theta = \pi/2$.

$\Rightarrow U$ is a rotation about the origin thru angle $\pi/2$ counterclockwise.

(OR: Compute



Observe U is rotation about origin thru $\pi/2$ counterclockwise)

It follows that T is a rotation about some point $\underline{c} \in \mathbb{R}^2$ thru the same angle.

To find $\underline{c}$, solve $T(\underline{x}) = \underline{x}$:- $\begin{pmatrix} 3-y \\ x+7 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

i.e. $\begin{matrix} 3-y=x \\ x+7=y \end{matrix} \rightsquigarrow \begin{matrix} x+y=3 \\ x-y=-7 \end{matrix}$

$\Rightarrow x=-2, y=5$, $\underline{c} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

So T is rotation about the point $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ thru angle $\pi/2$ counterclockwise.

Q6. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

(a) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ z \\ -x \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}}_{A.} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\det A \underset{\substack{\text{expanding} \\ \text{along 1st row}}}{=} 0 \cdot \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix}$

$= 1 \cdot \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 \cdot (0 - 1 \cdot (-1)) = +1.$

trace (A) = sum of diagonal entries of A $= 0+0+0 = 0.$

(b) $\det A = +1 \Rightarrow$ T is a rotation about some line L thru the origin, thru some angle θ.

$0 = \text{trace } A = 1 + 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \pm 2\pi/3.$

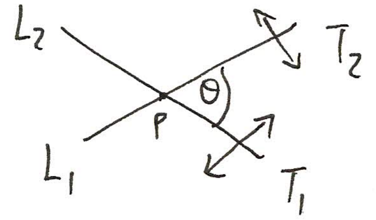
To find the direction of the line L (the axis of rotation), solve $T(\underline{x}) = \underline{x}$:-

$$\begin{pmatrix} -y \\ z \\ -x \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

So T is the rotation with axis the line L thru the origin in direction $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ thru angle $2\pi/3$

(direction of rotation not specified)
counterclockwise / clockwise.

Q7. (a)



In general, if L_1 & L_2 are two lines in $\mathbb{R}^2$ meeting at a point P, as shown, and T_1, T_2 are the reflections in L_1, L_2, then the composite $T_2 \circ T_1$ is a rotation about P thru angle 2θ counterclockwise

In our case $L_1 : \; x - 2y = -5$ $\quad *$

$L_2 : \; 2x + y = 10.$

Solving the equations simultaneously

$$\left(\text{e.g. } \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -5 \\ 10 \end{pmatrix} = \frac{1}{1-(-2)\cdot 2} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 10 \end{pmatrix} \right.$$
$$\left. = \frac{1}{5} \begin{pmatrix} 15 \\ 20 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)$$

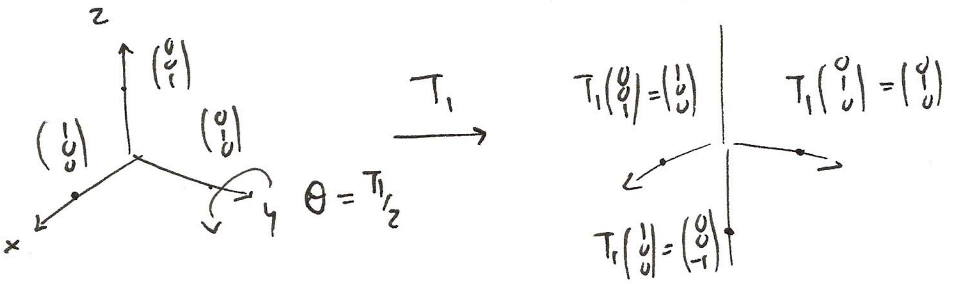
gives $P = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

To find the angle θ: The angle equals the angle between the normal vectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (given by the coefficients of x & y in the equations $*$)

But $\begin{pmatrix}1\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix} = 0$ so the lines are perpendicular, $\theta = \pi/2$.

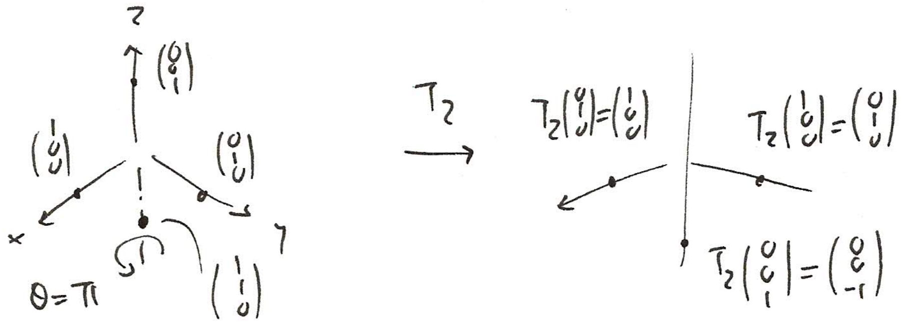
So $T_2 \circ T_1$ is rotation about $P = \begin{pmatrix}3\\4\end{pmatrix}$ thru angle $2\theta = \pi$.

(b) Write down matrices for T_1 & T_2 :-



w/ matrix

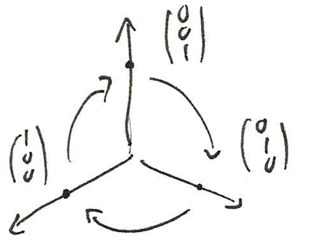
$$A_1 = \begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$$



w/ matrix

$$A_2 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}$$

Now $T_2 \circ T_1$ has matrix $A_2 \cdot A_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}\cdot\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$



$T_2 \circ T_1$

Observe $T_2 \circ T_1$ is rotation by thru angle $2\pi/3$ about axis L thru origin in direction $\begin{pmatrix}1\\1\\1\end{pmatrix}$, clockwise as viewed from $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

(Alternatively, compute as in Q6).