Math 462 Midterm, Wednesday 3/11/15, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 7 questions for a total of 70 points. Calculators, notes, and textbook are not allowed. Justify all your answers carefully.

Q1 (10 points). Let S^2 be a sphere in $\mathbb{R}^3$ with center the origin. Let $P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ be two points on S^2.

(a) (2 points) Compute the radius R of S^2.

(b) (4 points) Determine the equation of the great circle passing through P and Q.

(c) (4 points) Compute the length of the shortest path from P to Q along the surface of the sphere S^2, and describe the shortest path geometrically.

Q2 (10 points). Let S^2 denote the sphere in $\mathbb{R}^3$ with center the origin and radius 1. Let T be the spherical triangle on S^2 with vertices

$$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) (4 points) Compute the equations of the great circles given by the sides of T.

(b) (4 points) Compute the angles of T. [You may assume that each angle of T is less than or equal to $\pi/2$ radians.]

(c) (2 points) Deduce the area of T.

Q3 (9 points) Let S^2 be a sphere with radius R and C a spherical circle on S^2 with spherical radius r.

(a) (6 points) Determine a formula for the circumference of C in terms of R and r. [Justify your answer carefully.]

(b) (3 points) Now suppose r is small in comparison to R. Using the approximation $\sin(x) \approx x - x^3/6$ for small x, find an approximate value for the ratio A/B, where A is the circumference of C and B is the circumference of a circle in $\mathbb{R}^2$ of the same radius r.

Q4 (10 points). Give an algebraic formula $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for the following isometries of $\mathbb{R}^2$, where A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector.

(a) (4 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is rotation about the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ through angle π counterclockwise.

(b) (6 points) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = x + 3$ followed by translation through distance $5\sqrt{2}$ parallel to the line in the direction of increasing x. [T is a glide reflection.]

Q5 (10 points). Give a precise geometric description of the following isometries as a translation, rotation, reflection, or glide reflection.

(a) (5 points)

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 - x \\ y + 5 \end{pmatrix}.$$

(b) (5 points)

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 - y \\ x + 7 \end{pmatrix}.$$

Q6 (9 points). Consider the isometry T of $\mathbb{R}^3$ given by

$$T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ z \\ -x \end{pmatrix}.$$

(a) (4 points) Compute the determinant and the trace of the 3×3 orthogonal matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(b) (5 points) Give a precise geometric description of T. [If T is a rotation or rotary reflection the direction (counter-clockwise / clockwise) of rotation may be omitted.]

Q7 (12 points). Give a precise geometric description of the compositions $T_2 \circ T_1$ of the following isometries.

(a) (6 points) $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $x - 2y = -5$ and $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $2x + y = 10$.

(b) (6 points) $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the y-axis through angle $\pi/2$ counterclockwise (as viewed from the positive y-axis) and $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the line L in direction $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ through angle π.