Math 461 Final review problems

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(1) Let $P = \frac{1}{3}(1,2,2) \in S^2$ and $Q = \frac{1}{9}(4,4,7) \in S^2$. Find the length of the shortest path on the sphere S^2 from P to Q.

(2) Let $P = \frac{1}{3}(2,2,1) \in S^2$ and $Q = \frac{1}{7}(2,3,6) \in S^2$. Find the equation of the spherical line through P and Q.

(3) Let $L \subset S^2$ be the spherical line with equation $x + y + 2z = 0$ and $M \subset S^2$ the spherical line with equation $x + 2y + 3z = 0$.

(a) Find the two points of intersection of L and M.

(b) Find the angle between L and M.

(c) L and M divide the sphere into 4 regions. What are the areas of the regions?

(4) Let $P = \frac{1}{3}(2,1,2) \in S^2$ and let $L \subset S^2$ be the spherical line with equation $x + 2y + 3z = 0$. Find the equation of the spherical line M such that $P \in M$ and M is perpendicular to L.

(5) For a spherical triangle, suppose that two sides have equal lengths. Show that the two angles opposite the sides are equal.

[Hint: What is the spherical cosine rule?]

(6) Let T be the spherical triangle on S^2 with vertices $A = (1,0,0)$, $B = \frac{1}{\sqrt{2}}(0,1,1)$, and $C = (0,0,1)$.

(a) Compute the equations of the spherical lines given by the sides of T.

(b) Compute the angles of T. (You may assume that each angle of T is less than or equal to $\pi/2$ radians.)

(c) Deduce the area of T.

(7) (a) Suppose the sphere S^2 is tiled by congruent spherical triangles with angles $\pi/3$, $\pi/3$ and $2\pi/5$. How many triangles are there in the tiling?

(b) Suppose the sphere S^2 is tiled by 20 congruent spherical equilateral triangles. What are the angles of the triangles?

(8) Find the spherical center, spherical radius, and circumference of the spherical circle $C = \Pi \cap S^2$, where $\Pi \subset \mathbb{R}^3$ is the plane with equation $3x + 4y + 5z = 6$.

(9) Use the Gall–Peters projection (HW8Q7) to show that the area of a spherical disc of spherical radius r equals $2\pi(1 - \cos r)$.

(10) Find an algebraic formula for the reflection $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ in the plane Π with equation $x + 2y + 2z = 0$.

(11) Find an algebraic formula for the rotation about the y-axis through angle θ in the counterclockwise direction as viewed from the point $(0, 1, 0)$ looking towards the origin.

(12) For each of the following orthogonal matrices A, classify the isometry $T(\mathbf{x}) = A\mathbf{x}$ of $\mathbb{R}^3$ as either the identity, a reflection in a plane through the origin, a rotation about an axis through the origin, or a rotary reflection (a reflection in a plane through the origin followed by a rotation about the axis through the origin perpendicular to the plane). Give a precise geometric description in each case: for a reflection, give the plane of reflection; for a rotation, give the axis and angle of rotation; for a rotary reflection, give the plane of reflection and angle and axis of rotation (you may omit the sense (counter-clockwise or clockwise) of rotation in the case of a rotation or rotary reflection).

(a) $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(b) $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$.

(c) $A = \frac{1}{11}\begin{pmatrix} 9 & -6 & -2 \\ -6 & -7 & -6 \\ -2 & -6 & 9 \end{pmatrix}$.

(13) Compute the composition of a rotation about the x-axis through $\pi/2$ counterclockwise (as viewed from $(1, 0, 0)$ looking towards the origin) followed by a rotation about the z-axis through $\pi/2$ counterclockwise (as viewed from $(0, 0, 1)$ looking towards the origin). (Give a precise geometric description of the composition.)

(14) Let $T_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane Π_1 with equation $x + y + z = 0$ and let $T_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane Π_2 with equation $x + 2y + 3z = 0$. What is the composition $T = T_2 \circ T_1$? (Give a precise geometric description.)

(15) Recall the stereographic projection $F \colon S^2 \setminus \{N\} \to \mathbb{R}^2$ given by the algebraic formula

$$F(x, y, z) = \frac{1}{1 - z}(x, y)$$

with inverse $F^{-1} \colon \mathbb{R}^2 \to S^2 \setminus \{N\}$ given by

$$F^{-1}(u, v) = \frac{1}{u^2 + v^2 + 1}(2u, 2v, u^2 + v^2 - 1).$$

For each of the following planes Π, find the image of the spherical circle $C = \Pi \cap S^2$ under stereographic projection. (The image is either a line or a circle in the uv-plane. In the case of a line give its equation in the form $v = mu + c$, in the case of a circle give its center and radius.)

(a) $3x + 5y + 7z = 7$.

(b) $x + 3y + z = 2$.

(16) (a) What are the images of the spherical lines that pass through the antipodal points $(0, 1, 0)$ and $(0, -1, 0)$ under stereographic projection?

(b) What are the images of the spherical circles with center the point $(0, 1, 0)$ or $(0, -1, 0)$ under stereographic projection.

(c) Check that the curves from parts (a) and (b) meet at right angles.