Math 421 Midterm 2 review questions

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(1) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a continuous function. Let C be a smooth curve contained in U, with endpoints α and β, oriented from α to β. Recall the definition of the contour integral $\int_C f(z)dz$: We choose a parametrization of C, that is, a function $z\colon [a,b] \to \mathbb{C}$ defined on a closed interval $[a,b] \subset \mathbb{R}$ such that (i) the range of z equals C, (ii) the derivative $z'(t) = x'(t) + iy'(t)$ of $z(t) = x(t) + iy(t)$ exists for all $t \in [a,b]$ and is continuous, and (iii) as t increases from a to b the point $z(t)$ moves along C from α to β. Then we define

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)dt.$$

In each of the following cases, compute the contour integral $\int_C f(z)\,dz$ from first principles (that is, write down a parametrization of the curve C and use the definition of the contour integral).

(a) C is the line segment joining $1+i$ and $3+2i$, oriented from $1+i$ to $3+2i$, and $f(z) = \bar{z}$ (complex conjugate), that is, $f(x+iy) = x-iy$.

(b) C is the circle with center the point $2+i$ and radius 3, oriented counterclockwise, and $f(z) = \frac{1}{z-(2+i)}$.

(2) Recall the following bound for contour integrals: Let $U \subset \mathbb{C}$ be an open set, $f\colon U \to \mathbb{C}$ a continuous function, and C a smooth curve of finite length in U. Because f is continuous and C has finite length, the values $f(z)$ for $z \in C$ are bounded, that is, for some positive real number M we have $|f(z)| \leq M$ for all $z \in C$. Then we have the bound

$$\left| \int_C f(z)\,dz \right| \leq \text{length}(C) \cdot M.$$

(a) Let C be the semi-circle $\{z = x + iy \mid |z| = 2 \text{ and } y \geq 0\}$. Determine a bound for the contour integral

$$\int_C \frac{3z + 7i}{z^4 + 1}\, dz.$$

(b) Let C be the semi-circle $\{z = x + iy \mid |z| = 1 \text{ and } x \geq 0\}$. Determine a bound for the contour integral

$$\int_C e^{(z^2)}\, dz.$$

[Hint: If $w = u + iv$ what is $|e^w|$?]

(c) Let C_R be the circle with center the origin and radius R, oriented counterclockwise. Define a function

$$f(z) = \frac{z^5 + 3iz}{z^7 + 2z^3 + 4}.$$

i. Determine a bound $|f(z)| \leq M(R)$ for $z \in C_R$ and R sufficiently large, where $M(R)$ is a function of R.

ii. Show that $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$.

iii. Deduce that $\int_{C_R} f(z)\, dz = 0$ for R sufficiently large.
[Hint: How are the integrals over C_{R_1} and C_{R_2} related when both R_1 and R_2 are large? What is Cauchy's theorem? (See Q5.)]

(3) Recall the first part of the complex fundamental theorem of calculus: Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a function. Let C be a curve contained in U with end points α and β, oriented from α to β. Suppose that f has a complex antiderivative F, that is, a function $F\colon U \to \mathbb{C}$ such that F is complex differentiable and $F' = f$. Then

$$\int_C f(z)\, dz = F(\beta) - F(\alpha).$$

In particular, if C is a closed curve (that is, $\alpha = \beta$) then $\int_C f(z)dz = 0$. In each of the following cases, compute the contour integral $\int_C f(z)dz$.

(a) C a curve with endpoints 1 and i, oriented from 1 to i, and $f(z) = z^3 + 4iz + 5$.

(b) C a curve with endpoints 0 and $2i$, oriented from 0 to $2i$, and $f(z) = e^{3iz}$.

(c) C the semi-circle $\{z = x + iy \mid |z| = 2 \text{ and } x \geq 0\}$, oriented counterclockwise, and $f(z) = \frac{1}{z}$.

(d) C the semi-circle $\{z = x + iy \mid |z| = 2 \text{ and } x \leq 0\}$, oriented counterclockwise, and $f(z) = \frac{1}{z}$.

[Hint: Define a single-valued version of the complex logarithm which is complex differentiable on an open set U containing the curve C.]

(4) Recall the construction of antiderivatives by contour integration (the second part of the complex fundamental theorem of calculus): Let $U \subset \mathbb{C}$ be an open set and $f: U \to \mathbb{C}$ a complex differentiable function. Suppose that U is simply connected, that is, U is connected (any two points in U can be joined by a curve contained in U) and there are "no holes" (if C is a simple closed curve contained in U then the region bounded by C is also contained in U). Then f has an antiderivative $F: U \to \mathbb{C}$ defined as follows: Fix a point $\alpha \in U$. For each $z \in U$, choose a curve C_z in U with endpoints α and z, oriented from α to z, and define

$$F(z) = \int_{C_z} f(w)dw.$$

(This contour integral does not depend on the choice of the curve C_z by the assumption that U is simply connected and Cauchy's theorem.)

In each of the following cases, determine whether the given function f has an antiderivative. If an antiderivative does exist, either give an explicit formula for it or express it as above using contour integrals. If an antiderivative does not exist, explain carefully why not.

(a) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \sin((1+i)z)$.

(b) $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = 1/z^4$.

(c) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = e^{(z^2)}$.

(d) $f: \mathbb{C} \setminus \{2i\} \to \mathbb{C}$, $f(z) = \frac{1}{z-2i}$.

(e) $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = \frac{\cos z}{z}$.

[Hint: (d) If an antiderivative exists then the integral around any closed curve equals zero (see Q3). (e) What is Cauchy's integral formula? (See Q6.)]

(5) Recall Cauchy's theorem: Let $U \subset \mathbb{C}$ be an open set, $f: U \to \mathbb{C}$ a complex differentiable function, and C a simple closed curve in U such that the region bounded by C is also contained in U. Then we have $\int_C f(z)dz = 0$.

In each of the following cases, determine the contour integral $\int_C f(z)\,dz$.

(a) C any simple closed curve in $\mathbb{C}$, $f(z) = e^{3z}\cos(5z)\sin(z^2+1)$

(b) C the circle with center $1+i$ and radius 2, oriented counterclockwise, and $f(z) = \frac{1}{z^2+9}$.

[Hint: What is the domain of $\frac{1}{z^2+9}$?]

(c) C the circle with center $3+i$ and radius 2, oriented counterclockwise, and $f(z) = \operatorname{Log}(z)$ (the principal value of the complex logarithm).

[Hint: Where is $\operatorname{Log}(z)$ complex differentiable?]

(d) C the boundary of the square with vertices $\pm 1 \pm i$, oriented counterclockwise, and $f(z) = \frac{e^z \sin(z)}{z^3-8}$.

(6) Recall Cauchy's integral formula: Let $U \subset \mathbb{C}$ be an open set, $f: U \to \mathbb{C}$ a complex differentiable function, α a point in U, and C a simple closed curve in U, oriented counterclockwise, such that α lies inside C and the region bounded by C is contained in U. Then

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}\,dz.$$

In each of the following cases, determine the contour integral $\int_C g(z)\,dz$.

(a) C the circle with center the origin and radius 4, oriented counterclockwise, and $g(z) = \frac{e^{iz}}{z-\pi}$.

(b) C the circle with center i and radius 3, oriented counterclockwise, and $g(z) = \frac{z+1}{z^2-4z+13}$.

[Hint: Factor the denominator of $g(z)$ and so express $g(z)$ in the form $f(z)/(z-\alpha)$, where α is inside C and $f(z)$ is complex differentiable on an open set U containing C and the region it bounds. Now use Cauchy's integral formula. (It's also possible to use partial fractions, but this takes more work.)]

(c) C the circle with center 2 and radius 1, oriented counterclockwise, and $g(z) = \frac{\mathrm{Log}(z)}{z^3 - ez^2}$.

(d) C the boundary of the square with vertices $1, i, -1, -i$, oriented counterclockwise, and $g(z) = \frac{z^2+3}{z^5+z}$

(7) Recall Cauchy's generalized integral formula: With the same assumptions and notation as in Cauchy's integral formula above, for any positive integer n the nth complex derivative of f at α is defined and is given by

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}}\, dz.$$

In each of the following cases, determine the contour integral $\int_C g(z)\, dz$.

(a) C the circle with center the origin and radius 5, and $g(z) = \frac{\cos(z)}{(z-\pi)^3}$.

(b) C the circle with center $2i$ and radius 2, oriented counterclockwise, and $g(z) = \frac{3z+5}{(z^2+1)^2}$

[Hint: Factor the denominator of $g(z)$ and so express $g(z)$ in the form $f(z)/(z-\alpha)^k$ where k is a positive integer, α is inside C, and $f(z)$ is complex differentiable on an open set U containing C and the region it bounds.]

(8) Let C be the circle with center the origin and radius 3, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{e^{2z}}{(z-1)^k}\, dz$$

for each integer k (not necessarily positive).

[Hint: What is Cauchy's theorem? What is the generalized Cauchy integral formula?]

(9) Let C be the circle with center the origin and radius 2, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{e^z}{z^2-1}\,dz$$

[Hint: The function $\frac{e^z}{z^2-1}$ is not defined at two points α_1 and α_2 inside C. To compute the integral, divide the region bounded by C into two parts, each containing one of the points, and express the integral over C as a sum of two integrals. Now evaluate each integral using the Cauchy integral formula.]

(10) Recall the Gauss mean value theorem (a special case of the Cauchy integral formula): Let $U \subset \mathbb{C}$ be an open set, $f\colon U \to \mathbb{C}$ a complex differentiable function, $\alpha \in U$ a point, and r a positive real number such that the circle C with center α and radius r is contained in U and the disc bounded by C is also contained in U. Then

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{it})dt$$

In words, $f(\alpha)$ is the average value of f on the circle C. Writing $f = u + iv$, the same is true for u and v. In particular, it follows that u and v can't have a local max or a local min at any point α in their domain. So any critical point of u or v must be a saddle point.

In each of the following cases, (i) express the given function f in the form $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real valued functions of x and y, (ii) find the critical points of u, and (iii) check that they are saddle points.

(a) $f(z) = z^2 + 4iz + 5$
(b) $f(z) = z^3 - 3z$.

[Hint: Recall that the critical points of the 3 functions f, u, and v are the same. We can use the criterion from 233 for a critical point to be a saddle point: $\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - (\frac{\partial^2 u}{\partial x \partial y})^2 < 0$.]

(11) Recall Liouville's theorem: Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex differentiable function with domain the whole of $\mathbb{C}$. Suppose that f is bounded, that is, there is a positive real number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is constant.

(a) Show directly (without using Liouville's theorem) that each of the following functions either does not have domain the whole of $\mathbb{C}$ or is not bounded.

   i. $f(z) = \frac{1}{z^2+4}$.
   ii. $f(z) = \cos(z)$.
   iii. $f(z) = \frac{z^2}{z^4+1}$.
   iv. $f(z) = ze^{-z^2}$.

(b) Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a complex differentiable function such that there is a positive real number M such that $|f(z)| \geq M$ for all $z \in \mathbb{C}$. Prove that f is constant.

(c) Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a complex differentiable function such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Prove that f is constant.

[Hint: (b) Consider the function $g(z) = 1/f(z)$. (c) A continuous function $F\colon R \to \mathbb{R}$ on a closed and and bounded set $R \subset \mathbb{R}^2$ (e.g. a rectangle) is bounded.]

(12) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a complex differentiable function. Let α be a point in U and R a positive real number such that the open disc with center α and radius R is contained in U. Then f has a power series expansion about $z = \alpha$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \quad \text{for} \quad |z-\alpha| < R,$$

where the coefficients are given by

$$a_n = \frac{f^{(n)}(\alpha)}{n!}. \tag{$\dagger$}$$

For example, the function $f(z) = \frac{1}{1-z}$ has domain $U = \mathbb{C} \setminus \{1\}$ and has power series expansion about $z = 0$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad \text{for} \quad |z| < 1.$$

(a) Suppose $f: \mathbb{C} \to \mathbb{C}$ is a complex differentiable function with domain the whole of $\mathbb{C}$. Where is the power series expansion of f about $z = 0$ valid?

(b) Compute the power series expansion about $z = 0$ for (i) e^z and (ii) $\cos(z)$ using the formula (†).

(c) Let α and β be complex numbers, $\alpha \neq \beta$. Find the power series expansion of $\frac{1}{z-\beta}$ about $z = \alpha$. Where is this power series expansion valid?

[Hint: Write

$$\frac{1}{z-\beta} = \frac{1}{(z-\alpha)-(\beta-\alpha)} = \frac{-1}{\beta-\alpha} \cdot \frac{1}{1-\frac{z-\alpha}{\beta-\alpha}}$$

and use the power series expansion in the example above.]

(d) Compute the power series expansion of $\frac{1}{(1-z)^2}$ about $z = 0$ in two ways: (i) by differentiating the power series expansion for $\frac{1}{1-z}$, and (ii) by squaring the power series expansion of $\frac{1}{1-z}$.

[Hint: (ii) If $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are two power series which converge for $|z| < R$ then we can expand the product in the usual way, obtaining

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots$$

for $|z| < R$.]

(e) Compute the power series expansion for $\frac{1}{(z-1)(z-i)}$ about $z = 0$.

[Hint: Either use partial fractions, or multiply two power series together as in (d).]

(13) Let $a_n \in \mathbb{C}$ be a sequence of complex numbers and consider the power series

$$\sum_{n=0}^{\infty} a_n z^n.$$

Then there is a real number $R \geq 0$ or $R = \infty$ such that the power series converges for $|z| < R$ and diverges for $|z| > R$. The number R is called the *radius of convergence* of the power series. If the limit

$$L = \lim_{n \to \infty} |a_{n+1}/a_n|$$

exists, then

$$R = 1/L.$$

(Here we allow the cases $L = 0$ and $L = \infty$ and use the convention that $1/0 = \infty$ and $1/\infty = 0$.)

Determine the radius of convergence in the following cases.

(a) $\sum_{n=0}^{\infty} 2^n z^n$.
(b) $\sum_{n=0}^{\infty} n z^n$.
(c) $\sum_{n=0}^{\infty} \frac{3^n}{n!} z^n$.
(d) $\sum_{n=0}^{\infty} \binom{n+k}{k} z^n$, where k is a fixed positive integer, and

$$\binom{n+k}{k} = \frac{(n+k)!}{n!k!} = \frac{(n+1)(n+2)\cdots(n+k)}{k!}.$$

(14) Let a_n be the sequence of *Fibonacci numbers* defined by $a_0 = a_1 = 1$ and

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2.$$

The first few terms of the sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Now consider the power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n.$$

(a) Prove that when the series converges we have

$$f(z) = 1 + zf(z) + z^2 f(z),$$

and so $f(z) = 1/(1 - z - z^2)$.

(b) Determine the radius of convergence of the power series $f(z)$. [Hint: What is the domain of the function $1/(1 - z - z^2)$?]