Math 421 Midterm 1 review questions

Paul Hacking

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[The symbol $(\star)$ denotes harder problems.]

(1) Let $z \in \mathbb{C}$ and write $z = x + iy$ where $x, y \in \mathbb{R}$. What is the *complex conjugate* $\bar{z}$? What is the geometric interpretation of the mapping $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$? Show that $\overline{zw} = \bar{z}\bar{w}$ for all $z, w \in \mathbb{C}$.

(2) Give a precise geometric description of the mapping $f : \mathbb{C} \to \mathbb{C}$, $f(z) = (3 + 4i)z$.

(3) Compute $(1 + \sqrt{3}i)^{100}$.

(4) Let n be a positive integer. Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined by $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_0, a_1, \ldots, a_n$ are complex numbers and $a_n \neq 0$. We say f is a *polynomial of degree* n. What is the range of f? If b is a complex number, what are the possibilities for the number of solutions of the equation $f(z) = b$?

(5) What is the range of the complex exponential function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$?

(6) Let w and z be complex numbers. What can you say about w and z if $e^w = e^z$?

(7) Find all complex solutions of the following equations. Justify your answers carefully.

   (a) $z^2 + 5z + 7 = 0$
   (b) $z^3 + 4z = 0$.
   (c) $z^3 - 5z^2 + 4z + 10 = 0$.

(d) $z + \frac{5}{z} = 2$.

(e) $e^z = 1 + i$.

(f) $(\star)$ $\sin z = i$.

(g) $z^2 + (2+2i)z + i = 0$.

(h) $z^4 - 1 = 0$.

(i) $z^4 + 16 = 0$.

(j) $z^3 + i = 0$.

(8) Give a precise geometric description of the image $f(R)$ of the region $R \subset \mathbb{C}$ under the mapping $f\colon \mathbb{C} \to \mathbb{C}$ (include a sketch of the image).

(a) $f(z) = (-2+2i)z$, $R = \{z = x + iy \in \mathbb{C} \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$.

(b) $f(z) = z^5$, $R = \{z = x + iy \in \mathbb{C} \mid x^2 + y^2 \leq 4 \text{ and } 0 \leq y \leq x\}$.

(c) $f(z) = e^z$, $R = \{z = x + iy \in \mathbb{C} \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq \pi/4\}$.

(d) $f(z) = e^{iz}$, $R = \{z = x + iy \in \mathbb{C} \mid y \geq 0\}$.

(e) $f(z) = \mathrm{Log}(z)$, $R = \{z = x + iy \in \mathbb{C} \mid x > 0 \text{ and } y > 0\}$.

(9) True or False?

(a) $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$.

(b) $e^{-z} = 1/e^z$ for all $z \in \mathbb{C}$.

(c) For $z, w \in \mathbb{C}$, if $e^z = e^w$ then $z = w$.

(d) $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$ for all $z \in \mathbb{C}$.

(e) $|\sin(z)| \leq 1$ for all $z \in \mathbb{C}$.

(f) $\cos(z + 2\pi) = \cos(z)$ for all $z \in \mathbb{C}$.

(g) $\sin(z + \pi) = -\sin(z)$ for all $z \in \mathbb{C}$.

(h) For $z \in \mathbb{C}$, if $\sin(z) \in \mathbb{R}$ then $z \in \mathbb{R}$.

(i) $\cos(z)^2 + \sin(z)^2 = 1$ for all $z \in \mathbb{C}$.

(j) $e^{\mathrm{Log}\, z} = z$ for all $z \in \mathbb{C}$.

(k) $\mathrm{Log}(e^z) = z$ for all $z \in \mathbb{C}$.

(l) $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$ for all nonzero $z, w \in \mathbb{C}$.

(m) $\log(zw) = \log(z) + \log(w)$ for all nonzero $z, w \in \mathbb{C}$.

(n) ($\star$) For $z, w \in \mathbb{C}$, if $\cos(z) = \cos(w)$ then $z = \pm w + (2\pi)k$ for some choice of sign and integer k.

(10) Prove that $\sin(z + \pi/2) = \cos(z)$ for all complex numbers z.

(11) Let w and z be complex numbers. What is the definition of the multi-valued expression w^z? Compute all the values of $(-1)^i$. (Here as usual i denotes a square root of -1.)

(12) Consider the transformation $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, $f(z) = 1/z$. Let C be the circle with center $0 \in \mathbb{C}$ and radius 1. Let D be the circle with center $1 \in \mathbb{C}$ and radius 1.

(a) Compute the image of C under the transformation f.

(b) Compute the image of $D \setminus \{0\}$ under the transformation f.

(c) The circles C and D intersect at the point $e^{i\pi/3} = (1 + \sqrt{3}i)/2$ (why?). Show that the angle between the curves C and D at $e^{i\pi/3}$ is equal to the angle between the image curves $f(C)$ and $f(D)$ at $f(e^{i\pi/3})$.

[Hint: (b) Parametrize the circle $D \setminus \{0\}$ by the function $g\colon (-\pi, \pi) \to \mathbb{C}$, $g(\theta) = (1 + \cos\theta) + i \sin\theta$. Now compute the composition $f(g(\theta))$ and use it to determine $f(D \setminus \{0\})$.]

(13) Let $S \subset \mathbb{R}^3$ be the sphere with center the origin and radius 1. What is the definition of the stereographic projection $\bar{F}\colon S \to \mathbb{C} \cup \{\infty\}$? What is the image of the following regions of the sphere under stereographic projection?

(a) The northern hemisphere $H_n = \{(x, y, z) \in S \mid z > 0\}$.

(b) The southern hemisphere $H_s = \{(x, y, z) \in S \mid z < 0\}$.

(c) The equator $E = \{(x, y, z) \in S \mid z = 0\}$.

(d) The "eastern hemisphere" $H_e = \{(x, y, z) \in S \mid x > 0\}$.

(e) The "polar ice cap" $I = \{(x, y, z) \in S \mid z > \frac{3}{4}\}$.

(14) Let U be a subset of the complex plane and $f\colon U \to \mathbb{C}$ a function. What does it mean to say a function f is complex differentiable at a point $a \in U$? In each of the following cases, determine whether the function $f\colon \mathbb{C} \to \mathbb{C}$ is complex differentiable.

(a) $f(x+iy) = (2x+4y) + (3x+5y)i$.

(b) $f(x+iy) = (3x^2 - 2xy - 3y^2) + (x^2 + 6xy - y^2)i$.

(c) $f(x+iy) = (e^y \cos x) - (e^y \sin x)i$.

[Hint: A function $f(x,y) = (u(x,y), v(x,y))$ is real differentiable if the partial derivatives of u and v exist and are continuous. A function $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable if it is real differentiable (identifying $\mathbb{C} = \mathbb{R}^2$) and the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied.]

(15) Compute the image of the region $R = \{z = x+iy \in \mathbb{C} \mid x \geq 1\}$ under the mapping $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $f(z) = 1/z$.

[Hint: First observe that f is invertible with inverse $f^{-1} = f$. Now write $f(z) = w = u + iv$. Then $z = f^{-1}(w)$. Use this to write the condition $x \geq 1$ in terms of u and v, and so determine an inequality defining the image $f(R)$. Finally use the inequality to give a geometric description of $f(R)$ ("complete the square").]

(16) Consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = \cos(z)$. Write $f(z) = w = u + iv$. Let c be a real number.

(a) Show that for $c \neq 0$ the image of the horizontal line $y = c$ under the transformation f is an ellipse in the uv-plane given by the equation

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

where a, b are positive real numbers depending on c. (You should express a and b as functions of c.)

(b) What is the image of the x-axis (the horizontal line $y = 0$) under the transformation f?

(c) Suppose c is not an integer multiple of $\pi/2$. Show that the image of the line $x = c$ is one half of the hyperbola in the uv-plane given by the equation

$$\left(\frac{u}{a}\right)^2 - \left(\frac{v}{b}\right)^2 = 1$$

where a, b are positive real numbers depending on c. (You should express a and b as functions of c.)

(d) Let k be an integer. What is the image of the line $x = k(\pi/2)$ under the transformation f?

(e) Use parts (a)–(d) to draw a sketch of the image of the coordinate grid in the xy-plane under the transformation f.

[Hint: Recall the formula $\cos z = \cos x \cosh y - i \sin x \sinh y$. The hyperbola with $a = b = 1$ was discussed in HW3Q6. The general case is similar. (e) What can you say about the angles between the curves?]