Math 421 Homework 9

Paul Hacking

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(1) Consider the function

$$\phi\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \quad \phi(x,y) = \log(x^2+y^2).$$

Show that ϕ is harmonic.

(2) For each of the following harmonic functions $\phi\colon \mathbb{R}^2 \to \mathbb{R}$, find a *harmonic conjugate* ψ of ϕ, that is, a function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f\colon \mathbb{C} \to \mathbb{C}$, $f(x+iy) = \phi(x,y) + i\psi(x,y)$ is a complex differentiable function.

(a) $\phi(x,y) = x - xy$.

(b) $\phi(x,y) = e^y \cos x$.

(c) $\phi(x,y) = 3x^2y - y^3$.

[Hint: Use the Cauchy–Riemann equations to determine ψ.]

(3) Consider the function

$$f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad f(z) = \frac{1}{z}.$$

(a) Write f in the form $f(x+iy) = u(x,y) + iv(x,y)$ where $u\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ and $v\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$.

(b) Show that the level curves of u are the y-axis and the circles with center on the x-axis passing through the origin. (Here a *level curve* of the function $u = u(x,y)$ is a curve in $\mathbb{R}^2$ with equation $u(x,y) = c$ for some constant $c \in \mathbb{R}$.)

(c) Determine the level curves of v.

(d) Using your answers to parts (b) and (c), explain geometrically why the level curves of u and v are orthogonal.

[Hint: (b) For $c \neq 0$, clear denominators in the equation $u(x, y) = c$ and complete the square to obtain the equation of a circle.]

(4) Consider the function

$$\phi \colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \quad \phi(x,y) = \log(\sqrt{x^2 + y^2}).$$

The function ϕ is harmonic. In this question we will show that ϕ is *not* the real part of a complex differentiable function $f \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$.

(a) Suppose that there does exist a complex differentiable function f with real part ϕ, and write $f(x + iy) = \phi(x, y) + i\psi(x, y)$ for some $\psi \colon \mathbb{R}^2 \setminus \{(0,0\} \to \mathbb{R}$. Use the Cauchy–Riemann equations to compute the vector field $\mathbf{V} := \nabla\psi = (\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y})$.

(b) By the fundamental theorem of calculus, we have

$$\int_C \nabla\psi \cdot d\mathbf{x} = \psi(\beta) - \psi(\alpha)$$

for C a curve in $\mathbb{R}^2 \setminus \{(0,0)\}$ with endpoints α and β, oriented from α to β. In particular $\int_C \nabla\psi \cdot d\mathbf{x} = 0$ for C a closed curve. Show however that $\int_C \mathbf{V} \cdot d\mathbf{x} \neq 0$ for C the circle with center the origin and radius 1, oriented counterclockwise. This is a contradiction, so there does not exist a complex differentiable function f with real part ϕ.

[Hint: If $\mathbf{V} = (P, Q)$ and $\mathbf{x} \colon [a, b] \to \mathbb{R}^2$, $\mathbf{x}(t) = (x(t), y(t))$ is a parametrization of C, then

$$\int_C \mathbf{V} \cdot d\mathbf{x} = \int_C P dx + Q dy = \int_a^b P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\, dt.$$

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(5) For each of the following pairs of open sets $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$, find a complex differentiable function $f \colon U \to \mathbb{C}$ such that f is one-to-one and the range of f equals V.

(a)
$$U = \{z = x + iy \mid 0 < x < 1 \text{ and } 0 < y < 1\}$$
and
$$V = \{w = u + iv \mid -2 < u < 0 \text{ and } 0 < v < 2\}.$$

(b) $U = \{z = x+iy \mid x > 0 \text{ and } y > 0\}$ and $V = \{w = u+iv \mid v > 0\}$.

(c) $U = \{z = x + iy \mid 0 < y < \pi\}$ and $V = \{w = u + iv \mid v > 0\}$.

(d)
$$U = \{z = x + iy \mid |z| < 1 \text{ and } x > 0\}$$
and
$$V = \{w = u + iv \mid u < 0 \text{ and } -\pi/2 < v < \pi/2\}.$$

(e) $U = \{z \mid |z| < 1\}$ and $V = \{w = u + iv \mid v > 0\}$.

[Hints: (a)–(d) can be done with standard functions. For (e) use a linear fractional transformation.]

(6) For each of the following pairs of open sets $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$, find a complex differentiable function $f\colon U \to \mathbb{C}$ such that f is one-to-one and the range of f equals V.

(a) $U = \{z = x + iy \mid 0 < x < 1\}$ and $V = \{w = u + iv \mid u > 0\}$.

(b)
$$U = \{z = x + iy \mid x > 0,\ y > 0 \text{ and } |z| < 1\}$$
and
$$V = \{w = u + iv \mid 0 < u < 1 \text{ and } v > 0\}.$$

(c)
$$U = \{z \mid |z - 1| > 1 \text{ and } |z - 2| < 2\}$$
and
$$V = \{w = u + iv \mid 0 < u < 1\}.$$

(d)
$$U = \{z = x + iy \mid |z| < 1 \text{ and } x > 0\}$$
and
$$V = \{w = u + iv \mid v > 0\}.$$

[Hints: (a) Compare Q5(c). Here you need to combine with rotation and scaling. (b) Compare Q5(d). (c) Begin by applying a linear fractional transformation f such that $f(U)$ is “simpler” than U. For an LFT f and a circle $C \subset \mathbb{C}^2$, $f(C)$ is a line if C passes through the point p such that $f(p) = \infty$. (d) Begin by applying a LFT.]