Math 421 Homework 8

Paul Hacking

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(1) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a complex differentiable function. We say that f has an *isolated singularity* at a point $\alpha \in \mathbb{C}$ if $\alpha \notin U$ and there is an $r > 0$ such that, writing

$$D = \{z \in \mathbb{C} \mid |z - \alpha| < r\}$$

for the open disc with center α and radius r, we have $D \setminus \{\alpha\} \subset U$.

In this case $f(z)$ has a *Laurent series expansion about* $z = \alpha$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n \quad \text{for } z \in D \setminus \{\alpha\},$$

and the singularity is either a *removable singularity* (if $a_n = 0$ for all $n < 0$), a *pole of order* $m > 0$ (if $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$), or an *essential singularity* (if $a_n \neq 0$ for infinitely many $n < 0$).

The coefficient a_{-1} is called the *residue of* $f(z)$ *at* $z = \alpha$ and denoted $\operatorname{Res}_{z=\alpha} f(z)$. The residue is important because it can be used to compute contour integrals (Cauchy's residue theorem). In particular

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}_{z=\alpha} f(z)$$

for C a sufficiently small simple closed curve around α, oriented counterclockwise.

For each of the following functions $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$, (i) compute the Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ of $f(z)$ about $z = 0$; (ii) classify the singularity as a removable singularity, a pole of order m for some positive integer m (to be determined), or an essential singularity; and (iii) determine the residue $\operatorname{Res}_{z=0} f(z)$ of $f(z)$ at $z = 0$.

(a) $\frac{e^z}{z^4}$
(b) $\frac{\sin z - z}{z^3}$
(c) $\frac{1-\cos 2z}{z^5}$
(d) $\frac{1}{z^3(z-1)}$
(e) $\sin(\frac{1}{z})$.

[Hint: In each case, use the known power series expansions of e^z, $\sin z$, $\cos z$, and $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ to determine the Laurent series expansion of $f(z)$.]

(2) For each of the following functions $f(z)$, (i) determine the singularities of f (the points $\alpha \in \mathbb{C}$ where f is not complex differentiable) and (ii) compute the residue at each isolated singularity.

(a) $\frac{z}{z^2+1}$.
(b) $\frac{e^z}{z^2-4z+3}$.
(c) $\frac{\text{Log}(z)}{z-e}$.
(d) $\frac{z^3+1}{(z-i)^2}$.
(e) $\frac{e^{3z}}{(z-2)^5}$.
(f) $\frac{1}{z^5-4z^4+4z^3}$.

[Hint: If $f(z) = \frac{g(z)}{(z-\alpha)^m}$ where $\alpha \in \mathbb{C}$, m is a positive integer, $g(z)$ is complex differentiable at α and $g(\alpha) \neq 0$, then f has a pole of order m at α, and $\text{Res}_{z=\alpha} f(z) = g^{(m-1)}(\alpha)/(m-1)!$. In particular, if $m = 1$ then $\text{Res}_{z=\alpha} f(z) = g(\alpha)$, and if $m = 2$ then $\text{Res}_{z=\alpha} f(z) = g'(\alpha)$.]

(3) (a) Let $f(z) = \tan(z) = \frac{\sin(z)}{\cos(z)}$. Find the singularities of f and determine the residue at each singularity.

(b) Let C be the circle with center the origin and radius 2, oriented counterclockwise. Compute the contour integral $\int_C \tan z \, dz$.

[Hint: (a) Let $\alpha \in \mathbb{C}$ be a complex number. Suppose $f(z) = a(z)/b(z)$ where a and b are complex differentiable at α and $a(\alpha) \neq 0$ and $b(\alpha) = 0$, $b'(\alpha) \neq 0$. In class we showed using the power series expansions of a and b about $z = \alpha$ that (i) f has a simple pole (a pole of order $m = 1$) at α and (ii) $\text{Res}_{z=\alpha} = \frac{a(\alpha)}{b'(\alpha)}$. (b) What is Cauchy's residue theorem?]

(4) Let C be the circle with center the origin and radius 1, oriented counterclockwise. Compute the contour integral

$$\int_C z^3 e^{1/z}\, dz.$$

(5) Let C be the circle with center the origin and radius 2, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{z+1}{(z^2+1)(z^2+9)}\, dz.$$

(6) Let R be a positive real number. Let $C_{1,R} = [-R, R] \subset \mathbb{R} \subset \mathbb{C}$ and $C_{2,R}$ be the semicircle

$$C_{2,R} = \{z = x + iy \mid |z| = R \text{ and } y \geq 0\}.$$

Let C_R be the simple closed curve given by $C_R = C_{1,R} + C_{2,R}$, oriented counterclockwise. Let $f(z) = \frac{e^{iz}}{z^2+1}$.

(a) Assume $R > 1$. Compute the contour integral $\int_{C_R} f(z)\, dz$.

(b) Show that $\int_{C_{2,R}} f(z)\, dz \to 0$ as $R \to \infty$.

(c) Using parts (a) and (b) or otherwise, determine the integrals

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx$$

and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\, dx.$$

[Hints: (b) Find a bound $|f(z)| \leq M(R)$ for $z \in C_R$ using $|e^{u+iv}| = e^u$ for $u, v \in \mathbb{R}$. Deduce a bound $|\int_{C_R} f(z)dz| \leq B(R)$ and show that $B(R) \to 0$ as $R \to \infty$. (c) Relate the given real integrals to the complex integral $\int_{C_{1,R}} f(z)\, dz$ using $e^{ix} = \cos x + i \sin x$. Check your answer using the fact that $\sin x$ is an odd function.]

(7) (a) Does $\mathrm{Log}(z)$ (the principal value of the complex logarithm) have an isolated singularity at $z = 0$? Explain.

(b) Does

$$f(z) = \frac{1}{e^{1/z} - 1}$$

have an isolated singularity at $z = 0$? Explain.

(8) Let $U \subset \mathbb{C}$ be an open set and $f \colon U \to \mathbb{C}$ a complex differentiable function. Let $\alpha \in \mathbb{C}$ be a complex number. If f has a pole at α what kind of singularity does $\frac{1}{f}$ have at α? If f has an essential singularity at α what kind of singularity does $\frac{1}{f}$ have at α?

(9) (Optional) Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 4}\, dx.$$

[Hint: Consider $\int_{C_R} \frac{e^{iz}}{z^4+4}\, dz$ where C_R is the contour described in Q6.]