Math 421 Homework 7

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(1) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a complex differentiable function. Suppose that the circle C with center the origin and radius 3 is contained in U and the disc bounded by C is also contained in U. Suppose that $|f(z)| \leq 5$ for any point z on the circle C. Determine a bound $|f'(0)| \leq M$ for $f'(0)$ (where M is a positive real number).

[Hint: What is the generalized Cauchy integral formula? (See your class notes or Section 51 of the textbook.)]

(2) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a complex differentiable function. Suppose that the circle C with center the origin and radius 5 is contained in U and the disc bounded by C is also contained in U. Suppose that $|f(z)| \leq 7$ for any point z on the circle C. Determine a bound $|f^{(3)}(2i)| \leq M$ for $f^{(3)}(2i)$ (the third complex derivative of f evaluated at $z = 2i$).

(3) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a complex differentiable function. The generalized Cauchy integral formula shows that f has complex derivatives of all orders. That is, f can be differentiated (in the complex sense) as many times as we like. In this question, we will study examples showing that this does not work for real differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0. \\ x^2 & \text{if } x \geq 0. \end{cases}$$

Show carefully that f is differentiable.

(b) Show that the derivative f' is not differentiable. In other words, f is not twice differentiable.

(c) (Optional) Let n be a positive integer. Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ such that f can be differentiated n times but not $n+1$ times.

[Hint: (a) f is differentiable at any point $a \in \mathbb{R}$ such that $a \neq 0$ because 0 and x^2 are differentiable. So we only need to check differentiability at the point $a = 0$. This can be done using the definition of the derivative as a limit: compute the two "one-sided limits" where $h \to 0$ from the left or the right and check that they are equal (then the (two-sided) limit exists and is equal to the one-sided limits).]

(4) Let $U \subset \mathbb{C} = \mathbb{R}^2$ be an open set and $f: U \to \mathbb{C}$ a complex differentiable function. Write $f = u + iv$ where $u: U \to \mathbb{R}$ and $v: U \to \mathbb{R}$ are real differentiable functions. Recall the *Cauchy–Riemann* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and the expression

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

of the complex derivative of f in terms of the partial derivatives of u and v.

The generalized Cauchy integral formula shows that if f is complex differentiable then f' complex differentiable. Use this fact, the Cauchy–Riemann equations, and the symmetry of mixed partial derivatives

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

to write down a set of equations relating the second partial derivatives of u and v with respect to x and y.

[Hint: You should find that the $8 = 2 \times 4$ second partial derivatives of u and v with respect to x and y can be split into two sets of 4 so that the partial derivatives in each set agree up to sign.]

(5) Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(\alpha) = \int_0^{2\pi} \cos(\alpha x)\, dx.$$

(a) Compute the derivative $f'(\alpha)$ for $\alpha \neq 0$ in two ways:

i. By evaluating the integral as an explicit function of α and differentiating with respect to α.

ii. By "differentiating under the integral sign" (so that $f'(\alpha)$ is expressed as an integral).

(b) Check your two answers in part (a) agree by evaluating the integral found in (a)(ii) using integration by parts.

[Hint: In class we used differentiation under the integral sign to deduce the generalized Cauchy integral formula

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$$

from the Cauchy integral formula

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-\alpha}\, dz.$$

In this question we are studying differentiation under the integral sign in the real case: if $g = g(x, \alpha)$ is a continuous function of two real variables x and α such that the partial derivative $\frac{\partial g}{\partial \alpha}$ is also continuous, then

$$\frac{d}{d\alpha}\left(\int_a^b g\, dx\right) = \int_a^b \frac{\partial g}{\partial \alpha}\, dx$$

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(6) Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(\alpha) = \int_1^e x^\alpha dx.$$

(a) Compute the derivative $f'(\alpha)$ for $\alpha \neq -1$ in two ways:

i. By evaluating the integral as an explicit function of α and differentiating with respect to α.

ii. By "differentiating under the integral sign" (so that $f'(\alpha)$ is expressed as an integral).

(b) Check your two answers in part (a) agree by evaluating the integral found in (a)(ii) using integration by parts.

[Hint: To differentiate x^α with respect to α, first write x^α as $e^{\alpha \log x}$.]

(7) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex differentiable function. Write $f = u + iv$ where $u\colon \mathbb{R}^2 \to \mathbb{R}$ and $v\colon \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions. Suppose that there is a real number M such that $u(x, y) \leq M$ for all x and y. Prove that f is constant.

[Hint: What is Liouville's theorem? Consider the function $g(z) = e^{f(z)}$.]

(8) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex differentiable function. Suppose that there are positive real numbers M and r such that if $|z| \geq r$ then $|f(z)| \leq M|z|$.

(a) Use the generalized Cauchy integral formula to prove that the second complex derivative $f''(z)$ is identically equal to zero, that is, $f''(\alpha) = 0$ for all $\alpha \in \mathbb{C}$.

(b) Use part (a) to give an algebraic expression for $f(z)$ (involving some arbitrary constants).

(c) (Optional) Suppose more generally that $f\colon \mathbb{C} \to \mathbb{C}$ is a complex differentiable function and there are positive real numbers M and r and a positive integer n such that if $|z| \geq r$ then $|f(z)| \leq M|z|^n$. Determine an algebraic expression for $f(z)$.

[Hint: Generalize the proof of Liouville's theorem.]