Math 421 Homework 6

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(1) Let C be the circle in $\mathbb{C} = \mathbb{R}^2$ with center $1+i$ and radius 2, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{z}{z^2+4}\,dz.$$

[Hint: This can be done using partial fractions. Compare HW5Q7.]

(2) Let C be the circle with center the origin and radius 4, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{e^{iz}}{z-i}\,dz.$$

[Hint: What is the Cauchy integral formula?]

(3) Let C be the boundary of the square in $\mathbb{C} = \mathbb{R}^2$ with vertices $\pm 1 \pm i$, oriented counter clockwise. Compute the integral

$$\int_C \frac{\cos z}{z^3+9z}\,dz.$$

[Hint: Rewrite the integrand in the form $\frac{f(z)}{z-\alpha}$ where $f(z)$ is complex differentiable on C and inside C, and α is a point inside C. Now use Cauchy's integral formula.]

(4) Recall the Gauss Mean Value Theorem: Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a complex differentiable function. Let α be a point in U and C a circle with center α and radius r, where r is sufficiently small so

that the circle C and the disc bounded by C are contained in U. Then $f(\alpha)$ is equal to the average value of $f(z)$ for $z \in C$:

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{it})dt. \tag{$\dagger$}$$

Here we have parametrized the circle C at constant speed by

$$z\colon [0, 2\pi] \to C, \quad z(t) = \alpha + re^{it}.$$

Now let n be a positive integer, $f(z) = z^n$, and $\alpha = 0$. Show by direct computation that the equation ($\dagger$) holds.

(5) Let $U \subset \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a complex differentiable function. Write $f(x+iy) = u(x,y) + iv(x,y)$ where u and v are real-valued functions of x and y. Show carefully that the critical points of the 3 functions f, u, and v are the same.

[Hint: Recall that we say a point α is a critical point of a complex differentiable function $f(z)$ if $f'(\alpha) = 0$. And we say a point $\alpha = (a,b)$ is a critical point of real differentiable function $u(x,y)$ if $\frac{\partial u}{\partial x}(a,b) = \frac{\partial u}{\partial y}(a,b) = 0$. How is the complex derivative of $f = u + iv$ related to the partial derivatives of u and v? What are the Cauchy–Riemann equations?]

(6) Consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = (1+i)z^2$.

  (a) Express $f(z)$ in the form $f(x+iy) = u(x,y) + iv(x,y)$ where u and v are real valued functions of x and y.
  (b) Find the critical points of u.
  (c) Show directly that all the critical points of u are saddle points.

[Hint: To show that the critical points are saddle points, one can use the following criterion from 233: u has a saddle point at a critical point (a,b) if $D(a,b) < 0$ where

$$D(x,y) = \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$$

.]

(7) Consider the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \cos z$.

(a) Express $f(z) = \cos z$ in the form $f(x + iy) = u(x, y) + iv(x, y)$ where u and v are real valued functions of x and y.

(b) Find the critical points of u.

(c) Show directly that all the critical points of u are saddle points.

[Hint: (c) Recall that the hyperbolic sine and cosine functions are defined by $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$. By direct calculation, we find

$$\frac{d}{dx}(\sinh x) = \cosh x, \ \frac{d}{dx}(\cosh x) = \sinh x.$$

(No signs needed here!) To show that the critical points are saddle points, one can either argue geometrically using the known graphs of sine, cosine, hyperbolic sine, and hyperbolic cosine. Or, one can use the criterion from 233 (see Q6 above) together with the above derivative formulas.]

(8) Compute the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2 - 2x + 5} dx$$

using a contour integral in the complex plane $\mathbb{C} = \mathbb{R}^2$.

(9) Let α be a complex number and R a positive real number. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a complex differentiable function such that $f(z) = \alpha$ when $|z| = R$. What is $f(z)$ for $|z| < R$? Justify your answer carefully.