Math 421 Homework 5

Paul Hacking

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(1) Let C be a curve in the plane $\mathbb{C} = \mathbb{R}^2$ and $z\colon [a,b] \to \mathbb{C}$ a parametrization of C. That is, $z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are continuously differentiable functions of t, the range $z([a,b])$ of z equals C, and z is one-to-one (if $t_1 \neq t_2$ then $z(t_1) \neq z(t_2)$). Recall that the length of C can be computed using the formula

$$\text{length}(C) = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

In each of the following cases, write down a parametrization of the curve and use it to compute the length of the curve.

(a) C is the circle with center the origin and radius R.

(b) C is the arc of the curve given by the equation $y = x^{3/2}$ between the points $(1,1)$ and $(4,8)$.

(2) Recall that if z and w are complex numbers then $|z+w| \leq |z| + |w|$ ("the triangle inequality"). Show that

$$|z - w| \geq ||z| - |w||.$$

Here $|z|$ and $|w|$ denote the lengths of the complex numbers z and w, and $||z| - |w||$ denotes the absolute value of the real number $|z| - |w|$.

[Hint: The new inequality is given by two instances of the triangle inequality.]

(3) Let $U \subset \mathbb{C}$ be an open subset and $f\colon U \to \mathbb{C}$ a continuous function. Let C be an oriented curve in U. Suppose that M is a positive real

number such that $|f(z)| \le M$ for all $z \in C$. In class we showed that the contour integral $\int_C f(z)\,dz$ satisfies the following inequality

$$\left|\int_C f(z)\,dz\right| \le M \cdot \mathrm{length}(C).$$

(a) Let

$$f(z) = \frac{z^2+4}{z^4+3iz}$$

and let C be the circle with center the origin and radius R. Show that

$$|f(z)| \le \frac{R^2+4}{R^4-3R} \text{ for all } z \in C$$

if R is sufficiently large.

(b) Use part (a) to write down an inequality $|\int_C f(z)\,dz| \le B(R)$ for R sufficiently large, where $B(R)$ is a function of R (to be determined).

(c) Use the inequality from part (b) to show that $\int_C f(z)dz \to 0$ as $R \to \infty$.

[Hint for part (a): Use the inequalities $|z+w| \le |z|+|w|$ and $|z-w| \ge ||z|-|w||$ from Q2 and the equalities $|zw| = |z|\cdot|w|$ and $|z/w| = |z|/|w|$.]

(4) Recall the statement of *Green's theorem* from 233: Let $U \subset \mathbb{R}^2$ be an open subset of $\mathbb{R}^2$, and $p\colon U \to \mathbb{R}$ and $q\colon U \to \mathbb{R}$ two functions with continuous partial derivatives $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, $\frac{\partial q}{\partial x}$, and $\frac{\partial q}{\partial y}$. Let C be a simple closed curve in U, oriented counterclockwise, and R the region bounded by C. Assume R is contained in U. Then

$$\int_C p\,dx + q\,dy = \int_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\,dxdy.$$

(Here a *closed curve* is a curve whose end points are the same, so that the curve "closes up". And we say a closed curve is *simple* if there are no self-intersections.)

Use Green's theorem to compute the following integrals.

(a) $\int_C x\,dy$ where C is the circle with center the origin and radius 1, oriented counterclockwise.

(b) $\int_C 2xy\, dx + (x^2 - y^2)dy$, where C is any simple closed curve.

(c) $\int_C e^x \sin y\, dx + e^x \cos y\, dy$, where C is any simple closed curve.

(5) Let $f(z) = e^{(z^2)} \cos(2z) \sin(3z)$. Let C be a simple closed curve in $\mathbb{C} = \mathbb{R}^2$. What can you say about the integral $\int_C f(z)dz$? Justify your answer carefully.

[Hint: What is Cauchy's theorem (also known as the Cauchy–Goursat theorem)?]

(6) Let

$$f(z) = \frac{3z}{z^2 + 4z + 13}.$$

(a) What is the domain $U \subset \mathbb{C}$ where f is defined?

(b) Compute the contour integral $\int_C f(z)dz$ where C is the circle with center the origin and radius 3. Justify your answer carefully.

(7) Let

$$f(z) = \frac{1}{z^2 - iz}.$$

Let C be a simple closed curve in $\mathbb{C}$, oriented counterclockwise, and contained in the domain of f.

(a) Use partial fractions to write $f(z)$ as a sum of two simpler rational functions.

(b) Using part (a) or otherwise, show that $\int_C f(z)\, dz = 0$ if 0 and i are either both inside C or both outside C; $\int_C f(z)\, dz = -2\pi$ if 0 is inside C and i is outside C; and $\int_C f(z)\, dz = 2\pi$ if i is inside C and 0 is outside C.

[Hint: The following result was proved in class. Let $\alpha \in \mathbb{C}$ be a complex number. Let C be a simple closed curve in $\mathbb{C}$ oriented counterclockwise and not passing through α. Then $\int_C \frac{1}{z-\alpha} dz = 2\pi i$ if α is inside C and $\int_C \frac{1}{z-\alpha} dz = 0$ if α is outside C.]

(8) Prove that the function

$$f \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad f(z) = \frac{1}{z}$$

does not have a complex anti-derivative. That is, there is no function $F\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that F is complex differentiable and $F' = f$.

[Hint: Recall that if $U \subset \mathbb{C}$ is an open subset, $f\colon U \to \mathbb{C}$ is a continuous function, $F\colon U \to \mathbb{C}$ is a complex antiderivative of f, and C is a curve in U with end points α and β (oriented from α to β), then

$$\int_C f(z)\,dz = F(\beta) - F(\alpha).$$

In particular if $\alpha = \beta$ then $\int_C f(z)\,dz = 0$.]