Math 421 Homework 4

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(1) For each of the following functions f, determine whether f is complex differentiable and, if so, give a formula for its complex derivative.

(a) $f\colon \mathbb{C} \to \mathbb{C}$, $f(x+iy) = (x+2y) + i(3x+4y)$.

(b) $f\colon \mathbb{C} \to \mathbb{C}$, $f(x+iy) = (5x+7y) + i(-7x+5y)$.

(c) $f\colon \mathbb{C} \to \mathbb{C}$, $f(x+iy) = (2x^2 - 6xy - 2y^2) + i(3x^2 + 4xy - 3y^2)$.

(d) $f\colon \mathbb{C} \to \mathbb{C}$, $f(x+iy) = e^y \sin x + ie^y \cos x$.

(e) $U = \{z = x+iy \in \mathbb{C} \mid x > 0\}$, $f\colon U \to \mathbb{C}$,

$$f(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x).$$

[Hint: If $f = u+iv$ is complex differentiable then the *Cauchy–Riemann equations* $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied. Conversely, if the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous, and the Cauchy–Riemann equations are satisfied, then f is complex differentiable. If $f = u+iv$ is complex differentiable, then its complex derivative is given by $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.]

(2) Compute the complex derivatives of the following functions f.

(a) $f(z) = (3+2i)z^7 + 4iz^2 + (1+5i)$.

(b) $f(z) = \frac{z+3}{z^2+i}$.

(c) $f(z) = e^{(1+i)z}$.

(d) $f(z) = z\sin(2iz)$.

(e) $f(z) = \log(iz^2+3)$.

(3) Let $a, b, c, d \in \mathbb{C}$ be complex numbers such that $ad - bc \neq 0$. Consider the function

$$f(z) = \frac{az + b}{cz + d}.$$

Assuming $c \neq 0$, the domain U of f is given by $U = \mathbb{C} \setminus \{-d/c\}$. Compute the complex derivative f' of f and show that $f'(z) \neq 0$ for all $z \in U$.

(4) Let $\alpha \in \mathbb{C}$ be a complex number, and consider the multivalued function

$$f \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad f(z) = z^\alpha.$$

Show carefully that $f'(z) = \alpha z^{\alpha - 1}$.

[Hint: Recall that by definition $z^\alpha = e^{\alpha \log z}$ where $\log z$ is the multivalued complex logarithm. The complex derivative of $\log z$ is the single-valued function $\frac{1}{z}$. Now use the chain rule.]

(5) Let C be the line segment between the points $1+i$ and $3+2i$ in $\mathbb{C} = \mathbb{R}^2$, with the orientation given by the direction from $1+i$ to $3+2i$.

(a) Write down a parametrization $z \colon [a, b] \to \mathbb{C}$ of C for some $a, b \in \mathbb{R}$, $a < b$.

(b) Using part (a) compute from first principles the contour integral $\int_C z\, dz$. (Do not use an antiderivative.)

(c) Check your answer to part (b) by computing the contour integral using an antiderivative.

(6) Let C be the circle with center the origin and radius 1 in $\mathbb{C} = \mathbb{R}^2$, with the counterclockwise orientation.

(a) Write down a parametrization $z \colon [a, b] \to \mathbb{C}$ of C for some $a, b \in \mathbb{R}$, $a < b$.

(b) Using part (a) compute from first principles the contour integral $\int_C z\, dz$. (Do not use an antiderivative.)

(c) Check your answer to part (b) by computing the contour integral using an antiderivative.

(7) Let C be a smooth curve in the plane with endpoints 1 and $2i$, with the orientation given by the direction from 1 to $2i$. Compute the contour integral

$$\int_C (z^3 + 2iz + 3)\, dz.$$

(8) Let C be the circle with center the origin and radius R, with the counterclockwise orientation. In class we showed that

$$\int_C \frac{1}{z}\, dz = 2\pi i.$$

Show that for any integer $n \neq -1$, we have

$$\int_C z^n\, dz = 0.$$

[Hint: What is an antiderivative of z^n? Why is the case $n = -1$ special?]