Math 421 Homework 3

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(1) Compute the principal value $\operatorname{Log}(z)$ of the complex logarithm of the following complex numbers z. (Here

$$\operatorname{Log}(w) = \operatorname{Log}(se^{i\phi}) = \log(s) + i\phi$$

where the angular coordinate ϕ is chosen so that $-\pi < \phi \leq \pi$.)

(a) $3i$

(b) $1 - i$.

(c) $-2 + 2\sqrt{3}i$

(2) In class we explained how to construct a multivalued function $g(z) = w^z$ for $0 \neq w \in \mathbb{C}$: we define $w^z = e^{z \log w}$ where $\log w$ is the multivalued complex logarithm given by

$$\log w = \log(se^{i\phi}) = \log(s) + i(\phi + 2\pi k), \text{for any integer } k.$$

Determine all the possible values of the following complex powers w^z. (Write your answers in the polar form $w^z = te^{i\psi}$.)

(a) $1^{1/n}$, where n is a positive integer.

(b) $(-i)^i$.

(c) $(1 + i)^i$.

(3) Let $f \colon \mathbb{R} \to \mathbb{R}^2 = \mathbb{C}$ be the parametrization of the circle center the origin and radius 1 given by $f(t) = (x(t), y(t)) = (\cos(t), \sin(t)) = \cos(t) + i\sin(t)$. Sketch a graph of the imaginary part of the function

$$F \colon \mathbb{R} \to \mathbb{C}, \quad F(t) = \operatorname{Log}(f(t)) = \operatorname{Log}(x(t) + iy(t)).$$

This graph shows that $\mathrm{Log}(z)$ does not vary continuously as z crosses the negative real axis.

(4) Give an example of two nonzero complex numbers w_1, w_2 such that

$$\mathrm{Log}(w_1 w_2) \neq \mathrm{Log}(w_1) + \mathrm{Log}(w_2).$$

(5) Let $w = se^{i\phi} \in \mathbb{C} \setminus \{0\}$ and $z = x + iy \in \mathbb{C}$.

(a) Express the multivalued function w^z in polar coordinates $w^z = te^{i\psi}$, where the radial coordinate t and the angular coordinate ψ are expressed as multivalued functions of s, ϕ, x and y.

(b) Show that the length $|w^z| = t$ is single-valued if z is real, and in this case we have $|w^z| = |w|^z$. Show by example that $|w^z|$ is not single-valued in general.

(c) Show that all values of w^z are real if z is purely imaginary (that is, $z = iy$) and $|w| = 1$.

(6) Recall the definition of the hyperbolic cosine and hyperbolic sine functions:

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.$$

(a) Prove that $(\cosh t)^2 - (\sinh t)^2 = 1$ for all $t \in \mathbb{R}$.

(b) Consider the curve C in the xy-plane defined by the equation $x^2 - y^2 = 1$. Sketch the curve C, marking the intersection points with the x-axis and showing any asymptotes of the curve. The curve C is called a *hyperbola*.

[Hint: You should find that the curve C has two "pieces". (For example , what are the possible values of x for (x, y) a point on C?) The equation $x^2 - y^2 = 1$ can be rewritten $y = \pm\sqrt{x^2 - 1}$. What is an approximate value for y when $|x|$ is large? Alternatively, note that the equation $x^2 - y^2 = 1$ can be written $(x + y)(x - y) = 1$. Write $u = x + y$ and $v = x - y$, so that with respect to the new linear coordinates u and v the equation of the curve C is $uv = 1$, or $v = 1/u$. Now sketch the curve C with respect to the uv-axes, and use this to sketch C with respect to the xy-axes.]

(c) Recall that the circle with center the origin and radius 1 has equation $x^2 + y^2 = 1$ and can be parametrized by the function

$$f\colon \mathbb{R} \to \mathbb{R}^2, \quad f(t) = (x(t), y(t)) = (\cos t, \sin t).$$

Describe a similar parametrization of one piece of the hyperbola C using the hyperbolic cosine and sine functions.

(7) In class we described the stereographic projection from the sphere to the plane. In this question we will consider the simpler case of stereographic projection from the circle to the line: Let C be the circle with center the origin and radius 1 in $\mathbb{R}^2$, and $N = (0, 1) \in C$. We define the stereographic projection

$$G\colon C \setminus \{N\} \to \mathbb{R}$$

from the circle C to the real line $\mathbb{R}$ as follows: Identify the real line $\mathbb{R}$ with the x-axis in $\mathbb{R}^2$. For a point $P \in C$ such that $P \neq N$, define $G(P)$ to be the intersection point of the line through N and P with the x-axis. (Please draw a picture!)

(a) Use similar triangles to derive a formula for $G(x, y)$.

(b) Find a formula for the inverse

$$G^{-1}\colon \mathbb{R} \to C \setminus \{N\}, \quad G^{-1}(t) = (x(t), y(t)).$$

[Hint: (b) Write $G(x, y) = t$ and solve for x and y as functions of t using this equation together with the equation $x^2 + y^2 = 1$ of the circle C (eliminate y and solve for x, discarding one solution $x = 0$). Notice that geometrically this amounts to computing the intersection points of the line through N and $Q = (t, 0)$ with the circle C — one of these points is N and the other is $F^{-1}(Q)$.]