Math 421 Homework 2

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(1) Find all complex solutions of the equation $z^3 = 27i$. Draw a picture of the solutions in the plane $\mathbb{C} = \mathbb{R}^2$.

(2) Repeat Q1 for the equation $z^4 = (-8 + 8\sqrt{3}i)$.

(3) Recall the quadratic formula

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

On the last homework, we used this formula to find all complex solutions of $az^2 + bz + c = 0$ when a, b, c are real.

Now observe that the same formula can be used to find all complex solutions when a, b, c are complex — provided we can compute the two square roots of the complex number $b^2 - 4ac$. Use this observation to find all complex solutions of the equation

$$z^2 + (2 + 4i)z + (-3 + 2i) = 0.$$

(4) Let $z = a + bi$, $a, b \in \mathbb{R}$. Recall that we can compute the two complex square roots of z as follows: First write z in polar coordinates

$$z = r(\cos\theta + i\sin\theta).$$

Then the two square roots are

$$\pm\sqrt{r}(\cos(\theta/2) + i\sin(\theta/2)).$$

(a) Suppose for simplicity that $b \geq 0$, so that $0 \leq \theta \leq \pi$ and $\cos(\theta/2) \geq 0$, $\sin(\theta/2) \geq 0$. Use the formulae

$$\cos(\theta) = 2(\cos(\theta/2))^2 - 1 = 1 - 2(\sin(\theta/2))^2$$

to write an explicit formula in terms of a and b for the two square roots of $z = a + bi$. (Simplify your formula as much as possible.)

(b) Check your formula is correct by squaring and simplifying.

(c) Use your formula to compute the two complex square roots of $z = 5 + 12i$.

(5) Draw a precise picture of the image $f(R)$ of the region $R \subset \mathbb{C}$ under the complex mapping $f\colon \mathbb{C} \to \mathbb{C}$ in each of the following cases.

(a) $R = \{x + iy \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$, $f(z) = (1 + i)z$.

(b) $R = \{x + iy \mid x \geq 0 \text{ and } y \geq 0\}$, $f(z) = z^2$.

(c) $R = \{x + iy \mid -x \leq y \leq 0 \text{ and } \sqrt{x^2 + y^2} \leq 2\}$, $f(z) = z^3$.

(d) $R = \{x + iy \mid \pi/2 \leq y \leq 3\pi/2\}$, $f(z) = e^z$.

(e) $R = \{x + iy \mid 0 \leq x \leq y \text{ and } 1 \leq \sqrt{x^2 + y^2} \leq e\}$, $f(z) = \mathrm{Log}(z)$. Here $\mathrm{Log}(z)$ is the *principal value* of log which satisfies $\mathrm{Log}\, z = u + iv$, $u, v \in \mathbb{R}$, $-\pi < v \leq \pi$.

(f) $R = \{x + iy \mid x \geq 0,\ y \geq 0, \text{ and } 1 \leq \sqrt{x^2 + y^2} \leq 2\}$, $f(z) = 1/z$.

(6) Find all complex solutions of the equation $\sin(z) = 0$. Justify your answer carefully.

(7) Show that $\cos(z + \pi) = -\cos(z)$ for all complex numbers z.

(8) Consider the complex mapping $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = z + \frac{1}{z}$.

(a) For each $\alpha \in \mathbb{C}$ determine the number of solutions of the equation $f(z) = \alpha$. Deduce in particular that the range of f equals $\mathbb{C}$.

[Hint: Convert the equation into a quadratic equation and solve using the quadratic formula.]

(b) Now consider the mapping $g\colon \mathbb{C} \to \mathbb{C}$, $g(z) = \cos z$. Using part (a) or otherwise, show that the range of g equals $\mathbb{C}$.