Math 421 Homework 1

Paul Hacking

September 22, 2015

(1) Compute the following products of complex numbers. Express your answer in the form $x + yi$ where x and y are real numbers.

(a) $(2 + i)(5 + 3i)$

(b) $(3 - 4i)(1 + 2i)$

(c) $(7 + 5i)(3 + 2i)$

(d) $(a + bi)(a - bi)$ where a and b are real numbers.

(2) Compute the following quotients of complex numbers. Express your answer in the form $x + yi$ where x and y are real numbers.

(a) $\frac{3+4i}{2+i}$

(b) $\frac{7+i}{2+5i}$

(c) $\frac{1+i}{1-i}$

(3) Express the following complex numbers $z = x + iy$ in polar coordinates $z = r(\cos\theta + i\sin\theta)$. For parts (a) to (d) give a precise value for θ using known special values of sine and cosine ($\sin(\pi/6) = 1/2$, $\sin(\pi/4) = 1/\sqrt{2}$, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/6) = \sqrt{3}/2$, $\cos(\pi/4) = 1/\sqrt{2}$, $\cos(\pi/3) = 1/2$). For part (e) use the inverse tangent function $\tan^{-1}\colon \mathbb{R} \to (-\pi/2, \pi/2)$.

(a) $z = 1 + i$.

(b) $z = \sqrt{3} + i$.

(c) $z = 1 - i$.

(d) $z = -\sqrt{3} + i$.

(e) $z = 3 + 4i$.

(4) Find all complex solutions of the equation $f(z) = 0$ for the each of the following polynomials $f(z)$.

(a) $z^2 + 3z + 4$.

(b) $z^2 - 4z + 13$.

(c) $z^3 + 6z^2 + 10z$.

(d) $z^3 - 4z^2 + 6z - 4$.

[Hints: A quadratic equation $az^2 + bz + c = 0$ with real coefficients a, b, c can be solved using the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

noting that if the quantity $d := b^2 - 4ac$ is negative, then $\sqrt{d} = i\sqrt{|d|}$. (Here we write $|x|$ for the absolute value of a real number x.) To solve the cubic equations, first find a real solution α by inspection, then divide $f(z)$ by $(z - \alpha)$ to obtain a quadratic polynomial $g(z)$, and solve the quadratic equation $g(z) = 0$ as before. Then the solutions of $f(z) = 0$ are given by $z = \alpha$ and the solutions of $g(z) = 0$.]

(5) If a quadratic equation $z^2 + bz + c = 0$ with real coefficients b and c has two complex solutions $A \pm Bi$ (where A and B are real numbers), show that $b = -2A$ and $c = A^2 + B^2$.

[Hint: If $z^2 + bz + c = 0$ has solutions α_1 and α_2, then $z^2 + bz + c = (z - \alpha_1)(z - \alpha_2)$. Now expand the product and compare coefficients.]

(6) Recall the *binomial theorem*: for n a positive integer and a and b real or complex numbers, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

Here

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1}$$

is the number of ways of choosing a subset of k objects from a set of n objects. The notation $\binom{n}{k}$ is pronounced "n choose k".

(a) Let $z = x + iy$ be a complex number and n a positive integer. Use the binomial theorem to express z^n in the form $A + Bi$ where A and B are real numbers expressed as functions of x and y.

[Hint: Do $n = 2$ and $n = 3$ first to get the idea. Then do the general case using the observation that, for k an integer, $i^k = (-1)^l$ if $k = 2l$ is even and $i^k = (-1)^l \cdot i$ if $k = 2l + 1$ is odd.]

(b) Now write z in polar coordinates,

$$z = r(\cos\theta + i\sin\theta).$$

Express z^n in polar coordinates $z^n = s(\cos\phi + i\sin\phi)$, where s and ϕ are expressed as functions of r and θ.

Comparing with your answer to part (a) explains why it is much easier to solve the equation $z^n = c$ for a complex number c using polar coordinates r, θ instead of Cartesian coordinates x, y.

(7) Compute the complex number $(1+i)^{2015}$. Simplify your answer as much as possible.

[Hint: Use polar coordinates as in Q6(b).]

(8) In class we showed that all complex solutions to the equation $z^n = 1$ (where n is a positive integer) are given by

$$z = \cos(2\pi k/n) + i\sin(2\pi k/n)$$

for $k = 0, 1, 2, \ldots, n-1$. Equivalently,

$$z = 1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$$

where $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$. These complex numbers are called the "nth roots of unity" or "nth roots of 1". The symbol ζ is the greek letter zeta.

It follows that we have the factorization

$$z^n - 1 = (z-1)(z-\zeta)(z-\zeta^2)\cdots(z-\zeta^{n-1}) \qquad (\dagger)$$

of the polynomial $z^n - 1$ into linear factors corresponding to the solutions of the equation $z^n - 1 = 0$.

(a) Compute the factorization (†) explicitly from first principles in the cases $n = 2, 3, 4$. [Hints: $n = 2$ is easy. For $n = 3$, first find a real solution α of $z^3 - 1 = 0$. Then divide $z^3 - 1$ by $z - \alpha$, and factor the resulting quadratic polynomial. For $n = 4$, first factor $z^4 - 1$ into two quadratic polynomials using the "difference of two squares" identity $A^2 - B^2 = (A + B)(A - B)$ (note that $z^4 = (z^2)^2$). Then factor each of the quadratic polynomials.]

(b) (Bonus question) Repeat part (a) for $n = 6$ and $n = 8$.

(c) We can (partially) check equation (†) by comparing the coefficient of z^{n-1} on both sides.

i. First, explain why in the expansion of the product

$$(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

(where $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $a_0, a_1, \ldots, a_n$ are complex numbers) we have $a_n = 1$ and $a_{n-1} = -(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$. So, in particular, the coefficient of z^{n-1} on the right hand side of equation (†) equals $-(1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1})$.

ii. Second, explain geometrically why we have

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0$$

(Note that if $z_1, \ldots, z_n$ are complex numbers, regarded as points in the plane $\mathbb{R}^2$, then the average $(z_1 + z_2 + \cdots + z_n)/n$ is the center of mass of a collection of n particles of equal weight positioned at the points $z_1, z_2, \ldots, z_n$.)

(9) Consider the function

$$f\colon \mathbb{C} \to \mathbb{C}, \quad f(z) = z^2.$$

(a) Introduce notation $w = f(z) = u + iv$ and $z = x + iy$, where u, v, x, y are real numbers. Express u and v as functions of x and y.

(b) Consider the line L_1 in the xy-plane given by the equation $x = 1$. Find the equation of the image $f(L_1)$ in the uv-plane of the line L_1 under the transformation f. (Here as usual we are identifying $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x + iy$ and $w = u + iv$).]

(c) Repeat part (b) for the line L_2 in the xy-plane given by the equation $y = 1$.

(d) Show that the two curves found in parts (b) and (c) meet at two points in the uv-plane, and at each point the curves are perpendicular. (Recall that the angle between two curves meeting at a point is defined to be the angle between the tangent lines to the curves at the point.)

(e) Finally, explain why the image $f(S)$ of the square S in the xy-plane defined by

$$S = \{(x, y) \mid -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$$

under the transformation f is the region in the uv-plane bounded by the curves $f(L_1)$ and $f(L_2)$.

[Hints: In parts (b) and (c), the image is a curve in the uv-plane given by an equation $u = g(v)$ for some function g (to be determined). (d) The slope of the tangent line to $u = g(v)$ at a point $(u, v) = (a, b)$ is given $1/g'(b)$, where $g'(b)$ is the derivative of g evaluated at b. (Here the slope is $1/g'(b)$ instead of $g'(b)$ because the u-axis is horizontal and the v-axis is vertical.) And two lines are perpendicular if their slopes m_1, m_2 satisfy $m_1 m_2 = -1$. (e) Note that $f(z) = f(-z)$, and use the description of the transformation f in polar coordinates (as in Q6(b)). You can view a related picture at `virtualmathmuseum.org/ConformalMaps/square`.]