Math 421 Final exam review questions

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(1) Find all complex solutions of the following equations.

(a) $e^z + 1 - i = 0$.

(b) $z^3 + 8i = 0$.

(c) $z + \frac{1}{z} = i$.

(2) In each of the following cases, determine the image $f(R)$ of the region R under the function f.

(a) $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$, and

$$R = \{z = x + iy \mid 0 < x < 1 \text{ and } 0 < y < \pi/4\}.$$

(b) $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = \operatorname{Log} z$, and

$$R = \{z = x + iy \mid |z| > 1 \text{ and } y > 0\}.$$

(c) $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$, and

$$R = \{z = x + iy \mid |z| < 2,\ x < 0 \text{ and } x + y > 0\}.$$

(d) $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = 1/z$, and

$$R = \{z = x + iy \mid |z| < 1, x > 0 \text{ and } y > 0\}.$$

(3) Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ given by complex conjugation $f(z) = \bar{z}$, that is, $f(x + iy) = x - iy$.

(a) Show that f is *not* complex differentiable.

(b) Compute the contour integral $\int_C f(z)\,dz$ where C is the boundary of the triangle with vertices $0, 1$ and i, oriented counterclockwise.

(4) Compute the following contour integrals

(a)

$$\int_C z^3 + 3z + 5\,dz,$$

where C is a curve with endpoints 1 and i, oriented from 1 to i.

(b)

$$\int_C \cos(z^2)\,dz,$$

where C is the circle with center the origin and radius 1, oriented counterclockwise.

(c)

$$\int_C \frac{1}{(z-i)^3}\,dz$$

where C is the circle with center the origin and radius 2, oriented counterclockwise.

(5) Does the function

$$f\colon \mathbb{C} \to \mathbb{C}, \quad f(z) = e^{(z^2)} \sin(z^3)$$

have an antiderivative? Justify your answer.

(6) Compute the power series expansion of the function

$$f(z) = \frac{1}{z^2 - 5z + 6}$$

about $z = 0$. What is the radius of convergence of this power series?

(7) For each of the following functions $f(z)$, (i) compute the Laurent series expansion of the function about $z = 0$, (ii) classify the singularity $z = 0$ of the function as a removable singularity, a pole of order m for some positive integer m (to be determined), or an essential singularity, and (iii) compute the residue of $f(z)$ at $z = 0$.

(a)

$$\frac{\sin(z)}{z^4}.$$

(b)

$$\frac{e^z - 1 - z}{z^2}.$$

(c)

$$z\cos(1/z).$$

(d)

$$\frac{1}{z^4(z+1)}.$$

(8) The power series expansion for $\tan z$ about $z = 0$ is

$$\tan z = \frac{\sin z}{\cos z} = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$$

(where $\cdots$ denotes higher powers of z). Consider the function

$$f(z) = \frac{(z+1)\tan z}{z^5}.$$

(a) Given the power series expansion for $\tan z$ above, what is $\tan^{(5)}(0)$ (the 5th derivative of $\tan z$ evaluated at $z = 0$)?

(b) Compute the first few terms of the Laurent series expansion for $f(z)$ about $z = 0$.

(c) Classify the singularity of $f(z)$ at $z = 0$ as a removable singularity, a pole of order m for some positive integer m, or an essential singularity. What is the residue of $f(z)$ at $z = 0$?

(d) Determine the remaining singularities of $f(z)$ in the disc $D = \{z \mid |z| < 2\}$, classify their types, and compute the residue at each singularity.

(9) Let C be the circle with center the origin and radius 2, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{1}{z^2 + 2z - 3}\, dz.$$

(10) Let C be the circle with center the origin and radius 3, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{e^z}{z^3 - 2z^2}\, dz.$$

(11) Let C be the circle with center the origin and radius 1, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{\cos z}{e^z - 1}\, dz.$$

(12) (a) What is the Laurent series expansion of $e^{1/z}$ about $z = 0$?

(b) Let C be the circle with center the origin and radius 1, oriented counterclockwise. Compute the contour integral

$$\int_C \left(\frac{1}{z^2} + z + z^3\right) e^{1/z}\, dz.$$

(13) Let C be the circle with center the origin and radius 2, oriented counterclockwise. Compute the contour integral

$$\int_C \frac{\mathrm{Log}(z+3)}{(z-1)^n}\, dz$$

for any positive integer n.

(14) Compute the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx.$$

(15) Compute the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^4+1)} dx.$$

(16) What does it mean to say a function $\phi(x, y)$ is harmonic? For each of the following functions ϕ, (i) check that the function ϕ is harmonic and (ii) find a harmonic conjugate ψ of ϕ (that is, a function ψ such that $f = \phi + i\psi$ is complex differentiable).

(a) $4xy^3 - 4x^3y$.

(b) $e^{-x} \sin y$.

(17) Does there exist a complex differentiable function $f = u + iv$ with real part $u(x, y) = xe^y$?

(18) Let $f = u + iv$ be complex differentiable function with real part u and imaginary part v. Show that the vectors $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$ and $\nabla v = (\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y})$ are orthogonal at each point of the domain of f.

Now suppose two level curves $u = c$ and $v = d$ (where c and d are constants) intersect at a point p which is not a critical point of f. Explain why the curves are orthogonal at p.

(19) Give a precise geometric description of the transformation

$$f: \mathbb{C} \to \mathbb{C}, \quad f(z) = (1 + \sqrt{3}i)z.$$

(20) For each of the following pairs of open sets $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$, find a complex differentiable function $g: U \to V$ such that g is one-to-one and onto.

(a)
$$U = \{z = x + iy \mid 0 < y < 1\}$$
and
$$V = \{z = x + iy \mid x > 0 \text{ and } y > 0\}.$$

(b)
$$U = \{z = re^{i\theta} \mid r > 0 \text{ and } 0 < \theta < \pi/6\}$$
and
$$V = \{z = x + iy \mid y > 0\}.$$

(c)
$$U = \{z = x + iy \mid 0 < x < y\}$$
and
$$V = \{z = x + iy \mid y > 0\}.$$

(d)

$$U = \{z \mid |z - i| < 2\}$$

and

$$V = \{z = x + iy \mid y > 0\}.$$

(e)

$$U = \{z = x + iy \mid |z| < 1 \text{ and } x + y > 1\}$$

and

$$V = \{z = x + iy \mid 0 < y < x\}.$$

(f)

$$U = \{z = x + iy \mid |z| < 1 \text{ and } y > 0\}$$

and

$$V = \{z = x + iy \mid y > 0\}.$$