Math 421 Midterm 2, Wednesday 11/18/15, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 6 questions for a total of 60 points. Calculators, notes, and textbook are not allowed. Justify all your answers carefully. If you use a result proved in the textbook or class notes, state the result precisely.

Q1. (10 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be the function given by $f(z) = \bar{z}$, that is, $f(x+iy) = x-iy$. Let C be the line segment in $\mathbb{C}$ connecting 1 to i, oriented from 1 to i.

(a) (3 points) Show that f is *not* complex differentiable.

(b) (3 points) Write down a parametrization $z(t)$ of C.

(c) (4 points) Compute the contour integral $\int_C f(z)\,dz$.

Q2. (10 points) Compute the following contour integrals. Justify your answers carefully.

(a) (3 points)

$$\int_C (z^3 + iz + 3)\,dz,$$

where C is a smooth curve with endpoints 0 and $2i$, oriented from 0 to $2i$.

(b) (3 points)

$$\int_C e^{3z}\cos(z^2)\,dz,$$

where C is the circle with center the origin and radius 2, oriented counterclockwise.

(c) (4 points)

$$\int_C \frac{\mathrm{Log}(z)}{(z-2i)^2}\,dz,$$

where C is the circle with center $3i$ and radius 2, oriented counterclockwise.

Q3. (10 points) For each of the following functions f, determine whether there is a complex antiderivative F of f. In each case, either (i) describe an antiderivative by an explicit formula, or (ii) describe an antiderivative using a contour integral, or (iii) explain carefully why an antiderivative does not exist.

(a) (3 points) $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $f(z) = 1/z^2$.

(b) (3 points) $f\colon \mathbb{C}\setminus\{i\} \to \mathbb{C}$, $f(z) = 1/(z-i)$.

(c) (4 points) $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = \sin(z^2)$.

Q4. (10 points) Consider the function

$$f(z) = \frac{e^{iz}}{z^2 + 4}.$$

(a) (2 points) Determine the domain $U \subset \mathbb{C}$ of f.

(b) (2 points) State the Cauchy integral formula.

(c) (6 points) Let C be the circle with center $2 + i$ and radius 3, oriented counterclockwise. Compute the contour integral $\int_C f(z)\, dz$.

Q5. (10 points) Consider the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z + e^z$.

(a) (4 points) Find the critical points of f.

(b) (2 points) Express f in the form $f(x + iy) = u(x, y) + iv(x, y)$, where u and v are real valued functions of x and y.

(c) (4 points) Show that the critical points of u are saddle points.

Q6. (10 points) Consider the function

$$f(z) = \frac{z}{(z-1)(2z-1)}.$$

(a) (3 points) Express $f(z)$ in the form

$$f(z) = \frac{A}{z-1} + \frac{B}{2z-1}$$

for some constants A and B.

(b) (4 points) Using part (a) or otherwise, compute the power series expansion of $f(z)$ about $z = 0$.

(c) (3 points) Determine the radius of convergence of the power series expansion of $f(z)$ about $z = 0$.