Math 421 Midterm 1, Wednesday 10/14/15, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 7 questions for a total of 65 points. Calculators, notes, and textbook are not allowed. Justify all your answers carefully. If you use a result proved in the textbook or class notes, state the result precisely.

Q1. (6 points) Give a precise geometric description of the transformation $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = (1+2i)z$.

Q2. (8 points) Find all complex solutions of the equation $z^3 - 8i = 0$. Express the solutions in the form $z = x + iy$. Draw a picture of the solutions in the complex plane $\mathbb{C} = \mathbb{R}^2$.

Q3. (6 points) Find all complex solutions of the equation $e^{iz} + 7 = 0$.

Q4. (12 points) In each of the following cases, give a precise description of the image $f(R)$ of the region $R \subset \mathbb{C}$ under the transformation f. Include a sketch.

(a) (6 points) $R = \{z = x + iy \in \mathbb{C} \mid 0 \leq x \leq y \text{ and } x^2 + y^2 \leq 9\}$, $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^3$.

(b) (6 points) $R = \{z = x + iy \in \mathbb{C} \mid 0 \leq x \leq \pi \text{ and } \pi \leq y \leq 2\pi\}$, $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$.

Q5. (10 points) In each of the following cases, determine whether the function $f\colon \mathbb{C} \to \mathbb{C}$ is complex differentiable.

(a) (4 points) $f(x+iy) = (-4xy + 3y) + i(2x^2 - 4x - 2y^2)$.

(b) (6 points) $f(x+iy) = (xe^x \sin y + ye^x \cos y) + i(ye^x \sin y - xe^x \cos y)$.

Q6. (15 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be the transformation given by $f(z) = iz^2 + i$. Write $z = x + iy$ and $f(z) = w = u + iv$.

(a) (6 points) Let L_1 be the horizontal line with equation $y = 1$. Compute the image $f(L_1)$ of the line L_1 under the transformation f. (Find the equation in u and v defining the curve $f(L_1)$, and sketch the curve.)

(b) (6 points) Let L_2 be the vertical line with equation $x = 1$. Compute the image $f(L_2)$ of the line L_2 under the transformation f.

(c) (3 points) Determine the angle between the curves $f(L_1)$ and $f(L_2)$ at the point $f(1+i)$.

Q7. (8 points)

(a) (2 points) Explain geometrically why $|z+w| \leq |z| + |w|$ for all $z, w \in \mathbb{C}$.

(b) (6 points) Using part (a) or otherwise, prove that $|\cos z| \leq \cosh y$ for all $z = x + iy \in \mathbb{C}$.