Saturday 11/14/15 421 Midterm 2 review solutions.

1.a

$$z(t) = (1+i) + t\left((3+2i) - (1+i)\right)$$
$$= (1+i) + t(2+i)$$
$$= (1+2t) + i(1+t), \quad t \in [0,1]$$
$$z'(t) = 2+i$$
$$\int_C f(z)\,dz = \int_C \bar{z}\,dz = \int_0^1 \overline{z(t)}\, z'(t)\,dt$$
$$= \int_0^1 \left((1+2t) - i(1+t)\right)\cdot(2+i)\,dt$$
$$= \int_0^1 \left(2(1+2t) + (1+t)\right) + i\left((1+2t) - 2(1+t)\right)\,dt$$
$$= \int_0^1 (3+5t) + i(-1)\,dt$$
$$= \left[3t + 5t^2/2\right]_0^1 + i\left[-t\right]_0^1$$
$$= \boxed{11/2 - i}$$

b.

$$z(t) = 2+i+3e^{it}, \quad t \in [0, 2\pi]. \quad z'(t) = 3ie^{it}$$
$$\int_C f(z)\,dz = \int_C \frac{1}{z-(2+i)}\,dz = \int_0^{2\pi} \frac{1}{z(t)-(2+i)}\, z'(t)\,dt$$
$$= \int_0^{2\pi} \frac{1}{3e^{it}}\, 3ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

2a. $$\left|\frac{3z+7i}{z^4+1}\right| = \frac{|3z+7i|}{|z^4+1|}$$

$$|3z+7i| \leq |3z| + |7i| = 3|z|+7$$
$$|z^4+1| \geqslant \big||z^4| - |1|\big| = \big||z|^4-1\big|$$

$$\Rightarrow \left|\frac{3z+7i}{z^4+1}\right| \leq \frac{3|z|+7}{\big||z|^4-1\big|} \underset{\uparrow}{=} \frac{3\cdot 2+7}{2^4-1} = \frac{13}{15}$$

when $|z|=2$.

$$\therefore \left|\int_C \frac{3z+7i}{z^4+1}\,dz\right| \leq \text{length}(C)\cdot\frac{13}{15}$$
$$= \underbrace{\tfrac{1}{2}}_{\text{semi circle}}(2\pi\cdot \underbrace{2}_{\text{radius}})\cdot\frac{13}{15} = \frac{26\pi}{15}$$

b. $$|e^{z^2}| \underset{\uparrow}{=} |e^{(x^2-y^2)+i2xy}| = e^{x^2-y^2} \leq e^1$$

$z=x+iy$

for $z\in C$

(because $x^2-y^2\leq 1$ for $z=x+iy\in C$)

$$\Rightarrow \left|\int_C e^{(z^2)}\,dz\right| \leq \text{length}(C)\cdot e$$
$$= \tfrac{1}{2}(2\pi\cdot 1)\cdot e = \pi\cdot e.$$

c. i. $$|f(z)| = \frac{|z^5+3iz|}{|z^7+2z^3+4|} \leq \frac{|z|^5+3|z|}{\big||z^7| - |2z^3+4|\big|} \underset{\uparrow}{\leq} \frac{|z|^5+3|z|}{|z|^7-2|z|^3-4} = \frac{R^5+3R}{R^7-2R-4}$$

for $|z|=R$ sufficiently large $\dagger$

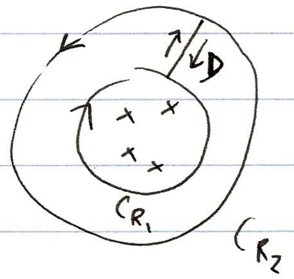
$\dagger$ explicitly, just need $R^7-2R-4>0$.

ii $$\left|\int_{C_R} f(z)\,dz\right| \le \text{length}(C_R)\cdot \frac{R^5+3R}{R^7-2R-4}$$
$$= 2\pi R \cdot \frac{(R^5+3R)}{R^7-2R-4}$$
$$= 2\pi \cdot \frac{(R^6+3R^2)}{R^7-2R-4}$$
$$= 2\pi\, \frac{(1/R + 3/R^5)}{1 - 2/R^6 - 4/R^7} \longrightarrow 2\pi \cdot \frac{0}{1} = 0$$
as $R \to \infty$.

$\therefore \int_{C_R} f(z)\,dz \longrightarrow 0$ as $R \to \infty$.

iii By Cauchy's theorem $\int_{C_{R_1}} f(z)\,dz = \int_{C_{R_2}} f(z)\,dz$

for R_1 & R_2 sufficiently large :-
The domain U of f is $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_n\}$
where $\alpha_1, \dots, \alpha_n$ are the zeroes of the denominator z^7+2z^3+4 of f. If R_1 & R_2 are large enough so that the region in between C_{R_1} & C_{R_2} is contained in U, then



$$0 \underset{\text{Cauchy's theorem}}{=} \int_{C_{R_2}+D-C_{R_1}-D} f(z)\,dz = \int_{C_{R_2}} f(z)\,dz - \int_{C_{R_1}} f(z)\,dz$$

Now by ii, $\int_{C_R} f(z)\,dz = \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$

for R sufficiently large.

3a. $\displaystyle\int_C z^3 + 4iz + 5 \, dz = \left[z^4/4 + 2iz^2 + 5z \right]_1^i$

$= (1/4 - 2i + 5i) - (1/4 + 2i + 5)$

$= -5 + i$

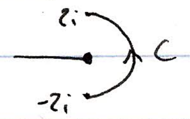
b. $\displaystyle\int_C e^{3iz} \, dz = \left[\frac{1}{3i} e^{3iz} \right]_0^{2i} = \frac{1}{3i}(e^{-6} - 1)$

$= -1/3 \, i (e^{-6} - 1)$

c. $\displaystyle\int_C \frac{1}{z} \, dz = \left[\operatorname{Log} z \right]_{-2i}^{2i}$

$= (\log 2 + \pi/2 \, i) - (\log 2 - \pi/2 \, i) = \pi i$

here, note that C is contained in $U = \mathbb{C} \setminus (-\infty, 0]$, the open set where $\operatorname{Log} z$ is complex differentiable.

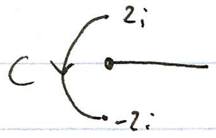


d. Define $F(z) = \log r + i\theta$ where $z = re^{i\theta}$ & $0 < \theta < 2\pi$.

$F : \mathbb{C} \setminus [0, \infty) \longrightarrow \mathbb{C}$

$F'(z) = 1/z$ on $U = \mathbb{C} \setminus [0, \infty)$.

$C \subset U \Rightarrow \displaystyle\int_C \frac{1}{z} \, dz = \left[F(z) \right]_{2i}^{-2i}$



$= \left(\log 2 + \frac{3\pi}{2} i\right) - \left(\log 2 + \frac{\pi}{2} i\right)$

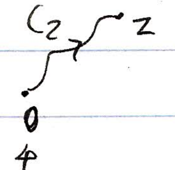
$= \pi i.$

4a. $f(z) = \sin((1+i)z)$. $F(z) = \dfrac{-1}{1+i} \cos((1+i)z)$ is an antiderivative

b. $f(z) = 1/z^4 = z^{-4}$. $\quad F(z) = \dfrac{z^{-3}}{-3} = \dfrac{-1}{3z^3}$

is an antiderivative.

c. $f(z) = e^{(z^2)}$. $\quad F(z) = \displaystyle\int_{C_z} e^{(z^2)}\, dz$



is an antiderivative

(note $U = \mathbb{C}$ is simply connected here.) (here we've chosen $\alpha = 0$)

d. If C is a simple closed curve such that $2i$ is inside C, then $\displaystyle\int_C \frac{1}{z-2i}\, dz = 2\pi i \neq 0$.

So $f(z) = \dfrac{1}{z-2i}$ does not have an antiderivative (see Ex 3). on $U = \mathbb{C} \setminus \{2i\}$.

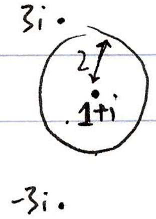
e. If C is a simple closed curve such that 0 is inside C, then
$$\int_C \frac{\cos z}{z}\, dz = 2\pi i \cdot \cos(0) = 2\pi i \neq 0$$
Cauchy's integral formula.

So $f(z) = \dfrac{\cos z}{z}$ does not have an antiderivative on $U = \mathbb{C} \setminus \{0\}$

5. a. $f(z) = e^{3z}\cos(5z)\sin(z^2+1)$
is complex differentiable on $\mathbb{C}$ (by the chain rule & product rule)
so $\displaystyle\int_C f(z)\, dz = 0$ for any simple closed curve C.

~~b. Log(z) is complex differentiable on $U = \mathbb{C} \setminus (-\infty, 0]$~~
~~and~~

b. $f : \mathbb{C} \setminus \{\pm 3i\} \longrightarrow \mathbb{C}$, complex differentiable.
The circle C and the disc bounded by C are contained in $U = \mathbb{C} \setminus \{\pm 3i\}$



because $|3i - (1+i)| = |-1+2i| = \sqrt{5} > 2$

and $|-3i - (1+i)| = |-1-4i| = \sqrt{17} > 2$

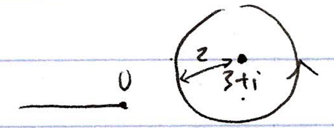
So $\int_C f(z)\,dz = 0$ by Cauchy's theorem

c. $\operatorname{Log} : \mathbb{C} \setminus (-\infty, 0] \longrightarrow \mathbb{C}$ complex differentiable.

C and the disc bounded by C are contained in U

(because $|z-(3+i)| \leq 2 \Rightarrow |x-3| \leq 2 \Rightarrow x \geq 1$

$z = x+iy \longrightarrow \;\; = \sqrt{(x-3)^2+(y-1)^2}$).

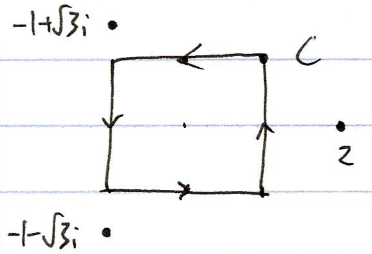


So $\int_C \operatorname{Log}(z)\,dz = 0$ by C.T.

d. $f(z) = \dfrac{e^z \sin z}{z^3-8}$

The domain U of f is $U = \mathbb{C} \setminus \underbrace{\{2, 2e^{i\frac{2\pi}{3}}, 2e^{i\frac{4\pi}{3}}\}}_{\text{zeros of } z^3-8}$

$= \mathbb{C} \setminus \{2, -1+\sqrt{3}i, -1-\sqrt{3}i\}$



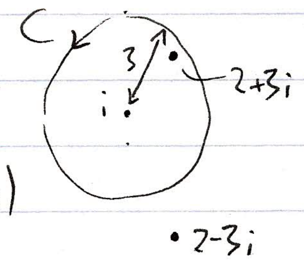
See the curve C and the square bounded by C are contained in U.

So $\int_C f(z)\,dz = 0$ by C.T.

6. a. $\int_C \dfrac{e^{iz}}{z-\pi}\,dz \underset{\text{CIF}}{=} 2\pi i \cdot e^{i\pi} = -2\pi i$

note π lies inside the circle C.

b. $\int_C \dfrac{z+1}{z^2-4z+13}\,dz$



$z^2-4z+13=0 \iff z = 2 \pm 3i$ (by quadratic formula)

$|2+3i-i| = |2+2i| = 2\sqrt{2} < 3$

$|2-3i-i| = |2-4i| = \sqrt{20} > 3$

$\Rightarrow 2+3i$ inside C, $2-3i$ outside C.

$$\Rightarrow \int_C \frac{z+1}{z^2-4z+13}\,dz = \int_C \frac{z+1}{(z-(2+3i))(z-(2-3i))}\,dz$$

$f(z)$

$$= \int_C \frac{\left(z+1/z-(2-3i)\right)}{z-(2+3i)}\,dz \underset{CIF}{=} 2\pi i \cdot f(2+3i)$$

$f(z)$ complex differentiable on C & inside C.

$$= 2\pi i\left(\frac{3+3i}{6i}\right) \quad \cancel{= \frac{1}{2}(1+i)}$$

$$= \pi(1+i).$$

c. $\dfrac{Log(z)}{z^3-ez^2} = \dfrac{Log(z)}{z^2\cdot(z-e)}$

complex differentiable on $U = \mathbb{C}\setminus\left((-\infty,0]\cup\{e\}\right)$

C circle center $2\in\mathbb{C}$, radius 1

$\Rightarrow$ e inside C $(e = 2.71..)$,

C & disc bounded by C contained in $\mathbb{C}\setminus(-\infty,0]$.

$$\therefore \int_C \frac{Log(z)}{z^3-ez^2}\,dz = \int_C \frac{(Log(z)/z^2)}{z-e}\,dz$$

$f(z)$, cx diff'ble on $\mathbb{C}\setminus(-\infty,0]$

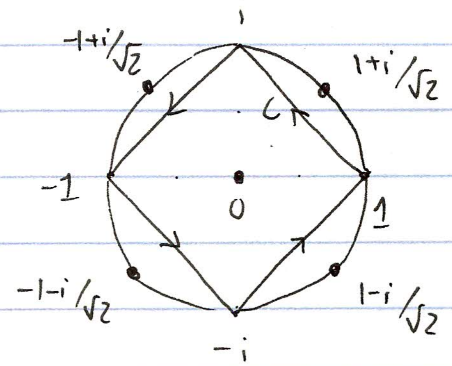
$$\underset{CIF}{=} 2\pi i\cdot f(e) = 2\pi i\cdot\frac{1}{e^2} = \frac{2\pi i}{e^2}.$$

d. $\dfrac{z^2+3}{z^5+z} = \dfrac{z^2+3}{z(z^4+1)}$

$z^5+z=0 \iff z=0$ OR $z^4=-1$

$\iff z=0,\ e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}$

$\iff z=0,\ (\pm1\pm i)/\sqrt{2}.$



0 is inside C, $\pm1\pm i/\sqrt{2}$ are outside C

$$\therefore \int_C \frac{z^2+3}{z^5+z} = \int_C \frac{\overbrace{\left(z^2+3/z^4+1\right)}^{f(z)}}{z}\,dz$$

$$\underset{\text{CIF}}{=} 2\pi i\cdot f(0) = 2\pi i\cdot 3/1$$

$$= 6\pi i$$

7. a) $$\int_C \frac{\cos(z)}{(z-\pi)^3}\,dz \underset{\substack{\text{GCIF,}\\ n=2}}{=} \frac{2\pi i}{2!} f^{(2)}(\pi)$$

(π is inside C ✓)

$f(z)=\cos z$ ($\Rightarrow f'(z) = -\sin z$, $f''(z) = -\cos z$)

$$= \pi i\cdot(-\cos\pi) = \pi i.$$

b) $$\int_C \frac{3z+5}{(z^2+1)^2}\,dz$$

$$\frac{3z+5}{(z^2+1)^2} = \frac{3z+5}{(z+i)^2(z-i)^2}$$ i inside C, $-i$ outside C

$$\therefore \int_C \frac{3z+5}{(z^2+1)^2}\,dz = \int_C \frac{\left(3z+5/(z+i)^2\right)}{(z-i)^2} = \frac{5\pi}{2}$$

$$\underset{\substack{\text{GCIF}\\ n=1,}}{=} \frac{2\pi i}{1!} f'(i) = 2\pi i\cdot\left(\frac{3(2i)^2-(3i+5)\cdot 4i}{(2i)^4}\right) = 2\pi i\cdot\left(\frac{-12+12-20i}{16}\right)$$

$f(z)= 3z+5/(z+i)^2$ ($\Rightarrow f'(z) = \dfrac{3(z+i)^2-(3z+5)2(z+i)}{(z+i)^4}$)

8. $$\int_C \frac{e^{2z}}{(z-1)^k}\,dz = \frac{2\pi i}{(k-1)!} f^{(k-1)}(1) = \frac{2\pi i}{(k-1)!} \cdot 2^{k-1} \cdot e^2$$

$$= \frac{2^k \cdot \pi \cdot e^2 \cdot i}{(k-1)!}$$

for $k \geq 1$

GCIF

$n = k-1$

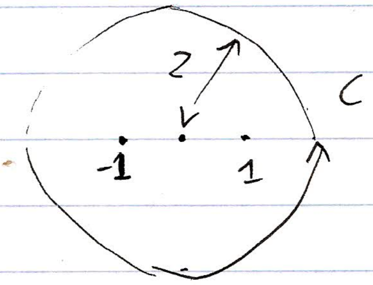
$f(z) = e^{2z}$

(Note: $f(z) = e^{2z} \Rightarrow f'(z) = 2e^{2z}$
$\Rightarrow \dots \Rightarrow f^{(k-1)}(z) = 2^{k-1} \cdot e^{2z}$)

If $k \leq 0$, $\frac{e^{2z}}{(z-1)^k} = e^{2z} \cdot (z-1)^{|k|}$ complex differentiable.

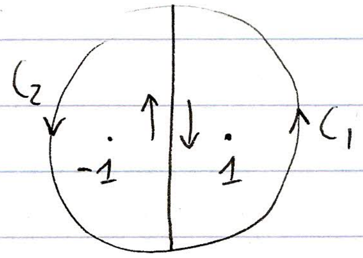
So $\int_C \frac{e^{2z}}{(z-1)^k}\,dz = 0$ by Cauchy's theorem.

9. $$\int_C \frac{e^z}{z^2-1}\,dz$$



$$\| $$

$$\int_{C_1} \frac{(e^z/z+1)}{z-1}\,dz + \int_{C_2} \frac{(e^z/z-1)}{z+1}\,dz$$



CIF ×2

$$= 2\pi i \cdot \frac{e^1}{1+1} + 2\pi i \cdot \frac{e^{-1}}{-1-1}$$

$$= \pi i \,(e - e^{-1}).$$

10 a.i. $f(z) = z^2 + 4iz + 5$

$$f(x+iy) = ((x^2-y^2) - 4y + 5) + i\,(2xy + 4x)$$
$$= u + iv.$$

ii. $u = x^2 - y^2 - 4y + 5.$

Critical points of u = critical points of f.

$f(z) = z^2 + 4iz + 5, \quad f'(z) = 2z + 4i = 0 \iff z = -2i$

$\therefore$ Critical point at $z = -2i$, i.e. $x = 0,\ y = -2$.

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y - 4$

iii. $\Rightarrow \frac{\partial^2 u}{\partial x^2} = 2 \quad \frac{\partial^2 u}{\partial y^2} = -2, \quad \frac{\partial^2 u}{\partial x \partial y} = 0.$

$$\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 < 0 \quad \Rightarrow \text{saddle.}$$

b. i. $f(z) = z^3 - 3z$

$f(x+iy) = \underbrace{(x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3)}_{\text{binomial theorem}} - 3(x+iy)$

$= (x^3 - 3xy^2 - 3x) + i(3x^2y - y^3 - 3y)$

$= u + iv$

ii. $f'(z) = 3z^2 - 3 = 0 \iff z = \pm 1.$

iii. $u = x^3 - 3xy^2 - 3x$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 3. \quad \frac{\partial u}{\partial y} = -6xy$

$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -6x, \quad \frac{\partial^2 u}{\partial x \partial y} = -6y$

$$\therefore \left.\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2\right|_{\substack{x=\pm 1\\ y=0}} = \left.\left(-(6x)^2 - (6y)^2\right)\right|_{\substack{x=\pm 1\\ y=0}} = -36 < 0.$$

$\Rightarrow$ saddle.

11 a. i. domain $= \mathbb{C} \setminus \{\pm 2i\}$ $\quad (z^2+4=0$ when $z = \pm 2i)$

ii. $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

So, e.g., $\cos(iy) = \cosh y \to \infty$ as $y \to \pm\infty$.

$\therefore$ not bounded.

iii.

domain $= \mathbb{C} \setminus \{ \pm 1 \pm i / \sqrt{2} \}$

iv

$$|f(z)| = |z| \cdot |e^{-z^2}|$$
$$= \sqrt{x^2+y^2} \cdot |e^{-(x^2-y^2) - 2ixy}|$$
$$= \sqrt{x^2+y^2} \cdot e^{-x^2+y^2}$$

So, e.g., $|f(iy)| = e^{y^2} \to \infty$ as $y \to \pm\infty$.

$\therefore$ not bounded.

b. $g(z) = 1/f(z)$, $g: \mathbb{C} \to \mathbb{C}$ cx differentiable

(because $|f(z)| \geq M > 0 \Rightarrow f(z) \neq 0$.)

Also $|g(z)| = 1/|f(z)| \leq 1/M$, g bounded.

L.T. $\Rightarrow$ $g(z)$ constant.

c. $R = \{ x+iy \in \mathbb{C} \mid 0 \leq x \leq 1, 0 \leq y \leq 1 \}$, a square.

$\subset \mathbb{C}$.

$|f(z)| \leq M$ for all $z \in R$, same $M \in \mathbb{R}$

(because f is continuous and R is closed and bounded)

Now for any $z \in \mathbb{C}$, we can write $z = w + (a+bi)$

where $w \in R$ and a & b are integers.

Then $f(z) = f(w+a+bi) = f(w)$ $\quad$ (because $f(z+1) = f(z)$, $f(z+i) = f(z)$ for any $z \in \mathbb{C}$)

So $|f(z)| = |f(w)| \leq M$.

So f bounded.

LT. $\Rightarrow$ f constant.

12. a. The power series expansion is valid for all $z \in \mathbb{C}$.

b. i.

$$f(z) = e^z \implies f^{(n)}(z) = e^z \text{ for all } n.$$

So $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n!} . z^n$

ii. $f(z) = \cos z \implies f'(z) = -\sin z$

$$f''(z) = -\cos z$$

$$f'''(z) = \sin z$$

$$f^{(4)}(z) = \cos z$$

$\vdots$ (repeats)

$$\implies f^{(n)}(z) = \begin{cases} \cos z & n = 4k \\ -\sin z & n = 4k+1 \\ -\cos z & n = 4k+2 \\ \sin z & n = 4k+3 \end{cases}$$ where k is an integer

$$\implies f^{(n)}(0) = \begin{cases} 1 & n = 4k \\ 0 & n = 4k+1 \\ -1 & n = 4k+2 \\ 0 & n = 4k+3. \end{cases}$$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} . z^n$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots = \sum_{m=0}^{\infty} (-1)^m . \frac{z^{2m}}{(2m)!}$$

c. $$\frac{1}{z-\beta} = \frac{1}{(z-\alpha)-(\beta-\alpha)} = \frac{-1}{(\beta-\alpha)} . \frac{1}{1-\left(\frac{z-\alpha}{\beta-\alpha}\right)}$$

$$= \frac{-1}{(\beta-\alpha)} \sum_{n=0}^{\infty} \left(\frac{z-\alpha}{\beta-\alpha}\right)^n = \sum_{n=0}^{\infty} \frac{-1}{(\beta-\alpha)^{n+1}} \cdot (z-\alpha)^n$$

valid for $\left|\frac{z-\alpha}{\beta-\alpha}\right| < 1$, i.e., $|z-\alpha| < |\beta-\alpha|$.

d. $$\frac{1}{1-z} = 1+z+z^2+\dots = \sum_{n=0}^{\infty} z^n \quad \text{for } |z|<1$$

$$\Rightarrow \frac{d}{dz}\left(\frac{1}{1-z}\right) = 1+2z+3z^2+\dots = \sum_{n=0}^{\infty}(n+1)\cdot z^n \quad \text{for } |z|<1$$

$$\parallel$$

$$\frac{1}{(1-z)^2} = -1\cdot\frac{-1}{(1-z)^2}$$

Alternatively, $$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)\cdot\left(\frac{1}{1-z}\right)$$

$$= (1+z+z^2+z^3+\dots)(1+z+z^2+z^3+\dots)$$

$$= 1+2z+3z^2+\dots$$

(the coefficient of z^n in the product is $\underbrace{(1\cdot1+1\cdot1+\dots+1\cdot1}_{n+1}) = n+1$)

e. $$\frac{1}{(z-1)(z-i)} = \frac{1}{(1-z)(i-z)} = \frac{1}{i\cdot(1-z)(1-z/i)}$$

$$= -i\cdot\left(\frac{1}{1-z}\right)\cdot\left(\frac{1}{1-z/i}\right) = -i\,(1+z+z^2+\dots)\left(1+z/i+(z/i)^2+\dots\right)$$

$$= -i\cdot(1+z+z^2+\dots)(1-iz-z^2+iz^3+z^4+\dots)$$

(repeats)

$$= -i\cdot\left(1+(1-i)z+(1-i-1)z^2+(1-i-1+i)z^3+\dots\right)$$

$$= -i\cdot(1+(1-i)z-iz^2+0\cdot z^3+z^4+(1-i)z^5+\dots)$$

(repeats)

$$= -i+(-1-i)z-z^2+0\cdot z^3-iz^4+(-1-i)z^5+\dots$$

(repeats)

OR $$\frac{1}{(z-1)(z-i)} = \frac{A}{z-1}+\frac{B}{z-i}, \quad A,B\in\mathbb{C}$$

$$1 = A(z-i) + B(z-1)$$
$$= (A+B)z \quad - Ai - B$$
$$\Rightarrow \quad A+B=0, \quad 1 = -Ai - B$$
$$B = -A, \quad 1 = A\cdot(1-i), \quad A = \frac{1}{1-i} = \frac{1+i}{2}, \quad B = \frac{-(1+i)}{2}.$$

Then $\dfrac{1}{(z-1)(z-i)} = \dfrac{1+i}{2}\left(\dfrac{1}{z-1} - \dfrac{1}{z-i}\right)$

$$= \frac{1+i}{2}\left(\frac{-1}{1-z} + \frac{1}{i}\cdot\frac{1}{1-z/i}\right)$$

$$= \frac{1+i}{2}\left(-(1+z+z^2+\dots) - i\cdot\left(1+z/i + (z/i)^2 + \dots\right)\right)$$

$$= -\frac{(1+i)}{2}\left((1+z+z^2+\dots) + (i + z - iz^2 - z^3 + iz^4 + \dots)\right)$$

repeats

$$= -\frac{(1+i)}{2}\left((1+i) + 2z + (1-i)z^2 + 0\cdot z^3 + (1+i)\cdot z^4 + \dots\right)$$

repeats

$$= -i - (1+i)z - z^2 + 0\cdot z^3 - iz^4 - \dots$$

repeats.

13\. a. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}}{2^n}\right| = \lim\limits_{n\to\infty} 2 = 2.$

$\Rightarrow \quad R = 1/2.$

b. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n+1}{n}\right| = \lim\limits_{n\to\infty} 1+\dfrac{1}{n} = 1.$

$\Rightarrow \quad R = 1/1 = 1.$

c. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{3^{n+1}/(n+1)!}{3^n/n!} = \lim\limits_{n\to\infty}\dfrac{3}{n+1} = 0$

$\Rightarrow \quad R = \text{"} 1/0 \text{"} = \infty.$

d. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+2)(n+3)\cdots(n+1+k)/k!}{(n+1)(n+2)\cdots(n+k)/k!} = \lim\limits_{n\to\infty}\dfrac{(n+1+k)}{(n+1)}$

$$= \lim_{n\to\infty} 1 + \frac{k}{n+1} = 1$$

$$\Rightarrow R = 1/1 = 1.$$

14. a. $1 + z f(z) + z^2 f(z)$

$$= 1 + z(a_0 + a_1 z + a_2 z^2 + \ldots) + z^2(a_0 + a_1 z + a_2 z^2 + \ldots)$$

$$= 1 + a_0 z + (a_1 + a_0) z^2 + (a_2 + a_1) z^3 + \ldots$$

$$= 1 + z + a_2 z^2 + a_3 z^3 + \ldots \quad \text{using } a_n = a_{n-1} + a_{n-2}$$

$$= f(z).$$

b. $f(z) = 1/{1-z-z^2}$ when convergent.

$$1 - z - z^2 = 0 \iff z^2 + z - 1 = 0,$$

$$z = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

$\therefore$ power series expansion for $\frac{1}{1-z-z^2}$ is convergent

for $|z| < R$, where $R = \frac{\sqrt{5}-1}{2}$

(note $\frac{\sqrt{5}-1}{2} = \left|\frac{\sqrt{5}-1}{2}\right| < \left|\frac{-1-\sqrt{5}}{2}\right|$)