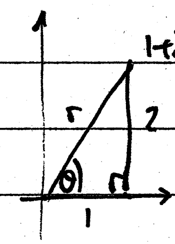
1. $f: \mathbb{C} \to \mathbb{C}$

$f(z) = (1+2i)\cdot z$

$1+2i = re^{i\theta} \qquad r = \sqrt{1^2+2^2} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1}(2)$



$\Rightarrow$ f is given by rotation about the origin thru angle $\theta = \tan^{-1} 2$ ccw followed by scaling by factor $r = \sqrt{5}$.

2. $z^3 - 8i = 0 \iff z^3 = 8i$

$z = re^{i\theta} \Rightarrow z^3 = r^3 e^{i3\theta} = 8i = 8e^{i\pi/2}$

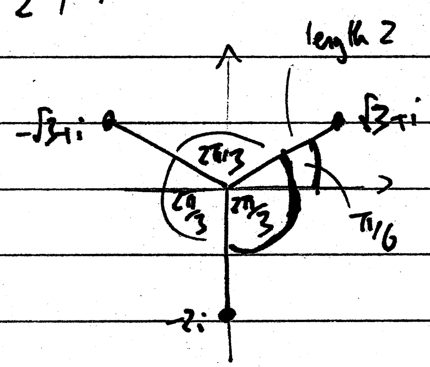
$\Rightarrow r^3 = 8, \quad 3\theta = \pi/2 + 2\pi k, \qquad k = 0,1,2$

$\Rightarrow r = 2, \quad \theta = \pi/6 + 2\pi k/3, \ k = 0,1,2$

$\therefore \ z = 2e^{i\pi/6}, \ 2e^{i5\pi/6}, \ 2e^{i3\pi/2}$

$= 2\cdot\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), \ 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), \ 2(-i)$

$= \sqrt{3}+i, \ -\sqrt{3}+i, \ -2i$



3. $e^{iz} + 7 = 0 \iff e^{iz} = -7$

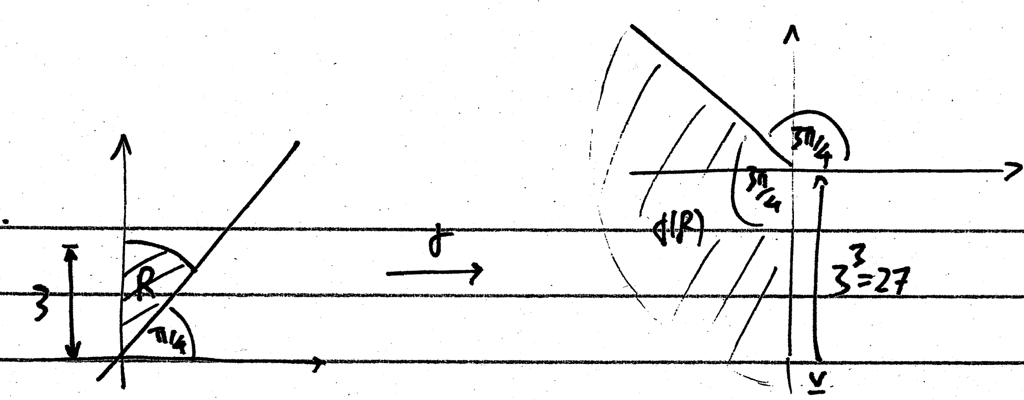
$\iff iz = \log(-7) = \log(7e^{i\pi}) = \log 7 + i(\pi + 2\pi k)$, k integer

multivalued complex logarithm

real logarithm

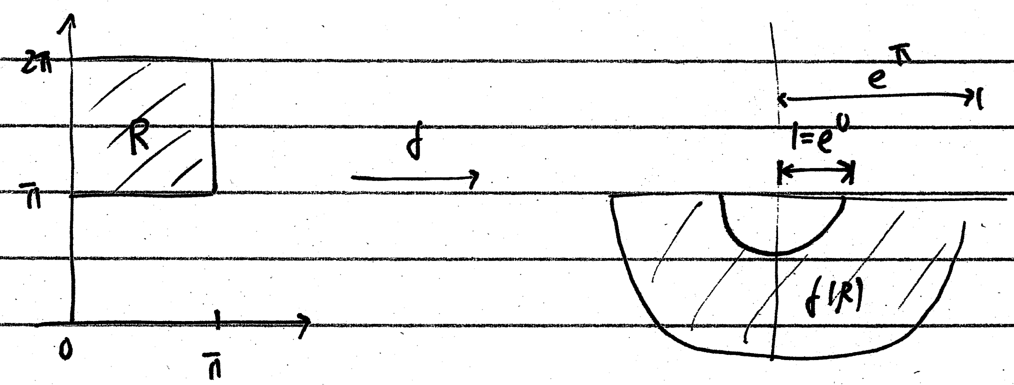
$\iff z = \pi + 2\pi k - i\log 7$, k integer (note $1/i = -i$)

4 a.



$f(z)=z^3$. $z=re^{i\theta} \Rightarrow z^3=r^3e^{i3\theta}$

b.



$f(z) = e^z = e^{x+iy} = se^{i\phi}$, $s = e^x$, $\phi = y$.

5. a $f(x+iy) = (-4xy+3y) + i(2x^2-4x-2y^2) = u+iv$.

If f is complex differentiable then the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ must be satisfied

But $\frac{\partial u}{\partial y} = -4x+3$, $\frac{\partial v}{\partial x} = 4x-4$, so $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$.

So f is NOT complex differentiable.

b. $f(x+iy) = (xe^x\sin y + ye^x\cos y) + i(ye^x\sin y - xe^x\cos y) = u+iv$

$\frac{\partial u}{\partial x} = e^x\sin y + xe^x\sin y + ye^x\cos y$

$\frac{\partial v}{\partial y} = e^x\sin y + ye^x\cos y - xe^x\cdot(-\sin y) = \frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial y} = xe^x\cos y + e^x \cos y + ye^x(-\sin y)$$

$$\frac{\partial v}{\partial x} = ye^x \sin y - e^x \cos y - xe^x \cos y = -\frac{\partial u}{\partial y}.$$

So the Cauchy-Rieman equations are satisfied

Also, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous

So f is complex differentiable.

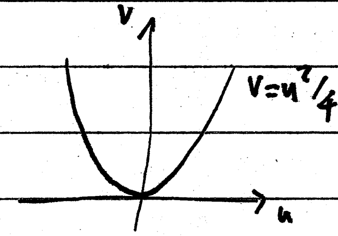
6. $f(z) = iz^2 + i$

$$f(x+iy) = i(x+iy)^2 + i = i(x^2 - y^2 + i \cdot 2xy) + i$$
$$= -2xy + i(x^2 - y^2 + 1) = u + iv.$$

a) $L_1 = (y = 1)$.

$y = 1 \Rightarrow u = -2x, \quad v = x^2 \Rightarrow v = u^2/4$

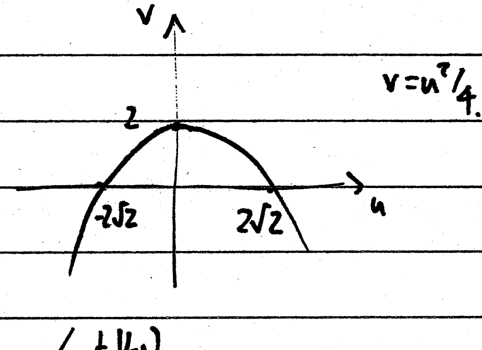
$\therefore f(L_1)$ is the parabola $v = u^2/4$



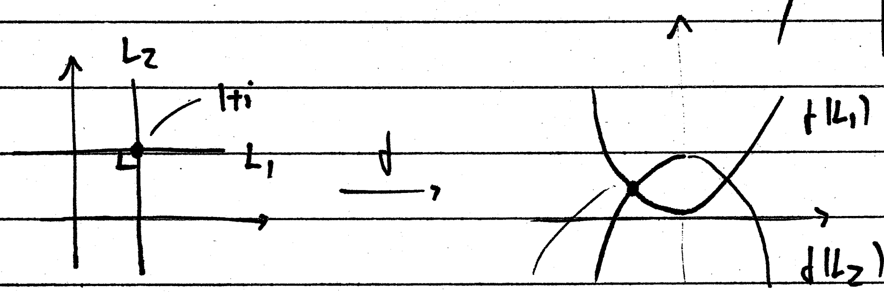
b) $L_2 = (x = 1)$

$x = 1 \Rightarrow u = -2y, \quad v = 2 - y^2 \Rightarrow v = 2 - u^2/4$

$\therefore f(L_2)$ is the parabola $v = 2 - u^2/4$



c)



$f(1+i) = -2+i$

f is complex differentiable : $f(z) = iz^2+i$

$$\Rightarrow f'(z) = 2iz$$

and $f'(1+i) \neq 0$

$\Rightarrow$ the angle between $f(L_1)$ and $f(L_2)$ at $f(1+i)$

is equal to the angle between L_1 and L_2 at $1+i$, which is $\pi/2$ rad.

Alternatively: Compute the slope of the tangent line to $f(L_1)$ & $f(L_2)$

at the point $(u,v) = (-2,1)$ $(= -2+i = f(1+i))$

using the equations $v = u^2/4$ of $f(L_1)$ & $v = 2 - u^2/4$ of $f(L_2)$

:-
$$m_1 = \left(\frac{u^2}{4}\right)'\Big|_{u=-2} = \frac{u}{2}\Big|_{u=-2} = -1$$

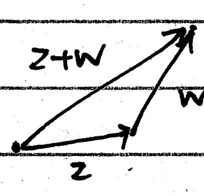
$$m_2 = \left(2 - \frac{u^2}{4}\right)'\Big|_{u=-2} = -\frac{u}{2}\Big|_{u=-2} = +1.$$

Since $m_1 \cdot m_2 = -1$ the tangent lines are perpendicular.

So the angle between the curves $f(L_1)$ and $f(L_2)$ at $f(1+i)$ is $\frac{\pi}{2}$ rad.

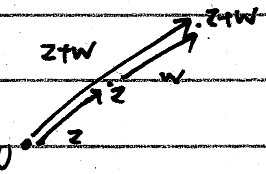
7. a. $|z+w| \leq |z| + |w|$ is the triangle inequality :

one side of a triangle has length less than or equal to the sum of the lengths of the other two sides (with equality if the triangle is degenerate) :-



vector addition in $\mathbb{R}^2$

(degenerate case:



)

This is just because the shortest path between two points is given by the line segment connecting the points.

b. $$|\cos z| = \left|\frac{e^{iz}+e^{-iz}}{2}\right| = \frac{1}{2}\,|e^{iz}+e^{-iz}|$$

$$\underset{a}{\leq} \frac{1}{2}\left(|e^{iz}| + |e^{-iz}|\right)$$

$$= \frac{1}{2}\left(|e^{i(x+iy)}| + |e^{-i(x+iy)}|\right)$$

$$= \frac{1}{2}\left(|e^{-y}\cdot e^{ix}| + |e^{y}\cdot e^{-ix}|\right)$$

$$= \frac{1}{2}\left(e^{-y}+e^{y}\right) = \cosh y. \quad \square$$