Monday 12/7/15 MATH 421 HW9 Solutions

1. $\phi = \log(x^2+y^2)$

$$\frac{\partial \phi}{\partial x} = \frac{2x}{x^2+y^2} \qquad \text{(chain rule)}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{2\cdot(x^2+y^2) - 2x\cdot 2x}{(x^2+y^2)^2} \qquad \text{(quotient rule)}$$

$$= \frac{2\cdot(y^2-x^2)}{(x^2+y^2)^2}$$

Similarly $\dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{2\cdot(x^2-y^2)}{(x^2+y^2)^2}$

So $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0.$

2. a. $\phi = x - xy$

$f = \phi + i\psi$ Cx diff'ble

$\Rightarrow$ CR eqs 1. $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$

2. $\dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}.$

i.e. 1. $\dfrac{\partial \psi}{\partial y} = \dfrac{\partial \phi}{\partial x} = 1-y \Rightarrow \psi = y - \tfrac{1}{2}y^2 + a(x)$

2. $\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \phi}{\partial y} = x \Rightarrow a'(x) = x \Rightarrow a(x) = \tfrac{x^2}{2} + c$

$\therefore \psi = y + \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2$ is a harmonic conjugate of ϕ.

b. $\dfrac{\partial \psi}{\partial y} = \dfrac{\partial \phi}{\partial x} = -e^y \sin x \Rightarrow \psi = -e^y \sin x + a(x)$

$\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \phi}{\partial y} = -e^y \cos x \Rightarrow a'(x) = 0 \Rightarrow a(x) = c.$

$\therefore \psi = -e^y \sin x$ is a harmonic conjugate of ϕ.

c $\phi = 3x^2y - y^3$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 6xy \implies \psi = 3xy^2 + a(x)$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -3x^2 + 3y^2 \implies a'(x) = -3x^2 \implies a(x) = -x^3 + c$$

$\therefore\ \psi = 3xy^2 - x^3$ is a harmonic conjugate of ϕ.

3. a. $$f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u} + i\underbrace{\left(\frac{-y}{x^2+y^2}\right)}_{v}$$

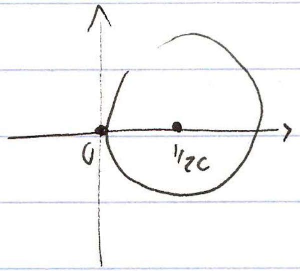
b. $u = c.$

$\iff x = c\cdot(x^2+y^2)$

If $c=0$: $x=0$. y-axis

$c \neq 0$ $x^2+y^2 - \frac{1}{c}x = 0.$

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \frac{1}{4c^2}$$



circle, center $(\frac{1}{2c}, 0)$, radius $\frac{1}{2c}$.

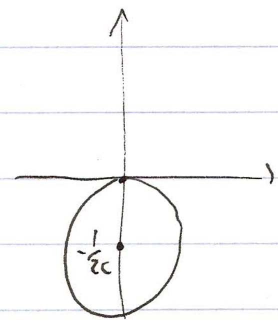
c. $v = c$

$\iff -y = c\cdot(x^2+y^2)$

$c=0$: $y=0$, x-axis.

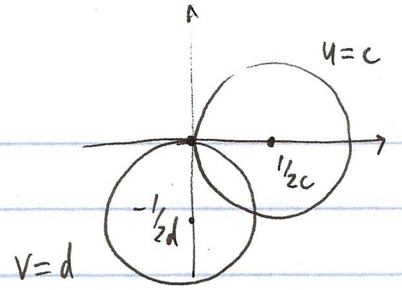
$c\neq 0$: $x^2+y^2+\frac{1}{c}y = 0$

$$x^2 + \left(y+\frac{1}{2c}\right)^2 = \frac{1}{4c^2}$$



circle, center $(0, -\frac{1}{2c})$, radius $\frac{1}{2c}$

d.



The curves meet at right angles at the origin (because they are tangent to the x and y-axes). Therefore by symmetry (reflecting in the line joining the centers of the two circles) they meet at right angles at the other intersection point too.

4. a

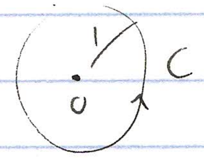
$$\underline{v} = \nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) \overset{CR}{=} \left(\frac{-\partial\phi}{\partial y}, \frac{\partial\phi}{\partial x}\right)$$

$$= \left(\frac{-\frac{1}{2}\cdot 2y\cdot(x^2+y^2)^{-1/2}}{(x^2+y^2)^{1/2}}, \frac{\frac{1}{2}\cdot 2x\cdot(x^2+y^2)^{-1/2}}{(x^2+y^2)^{1/2}}\right)$$

$$= \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

b.

$$\int_C \underline{v}\cdot d\underline{x} = \int_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$



$$= \int_0^{2\pi}\left(\frac{(-\sin t)}{1}\cdot(-\sin t) + \frac{\cos t}{1}\cdot\cos t\right)dt$$

$\underline{x}:[0,2\pi]\longrightarrow\mathbb{R}^2$

$\underline{x}(t) = (\cos t, \sin t)$

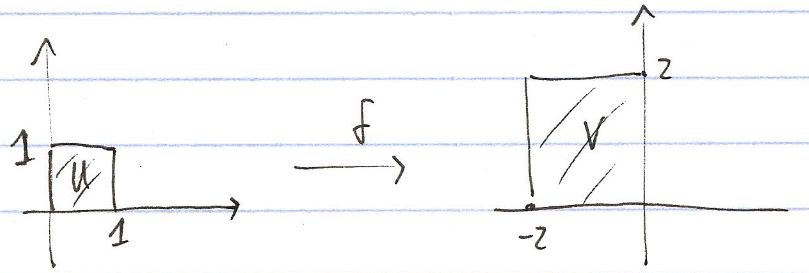
parametrization of C

$\underline{x}'(t) = (-\sin t, \cos t)$

$$= \int_0^{2\pi} (\sin t)^2 + (\cos t)^2\,dt$$

$$= \int_0^{2\pi} dt = 2\pi \neq 0.$$

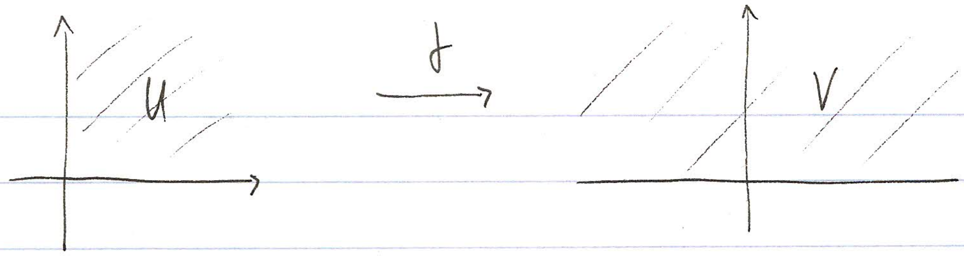
5. a.



$e^{i\pi/2} = i \Rightarrow$

$f(z) = 2iz$:—

scaling by 2, & rotation by $\pi/2$ ccw about the origin b.

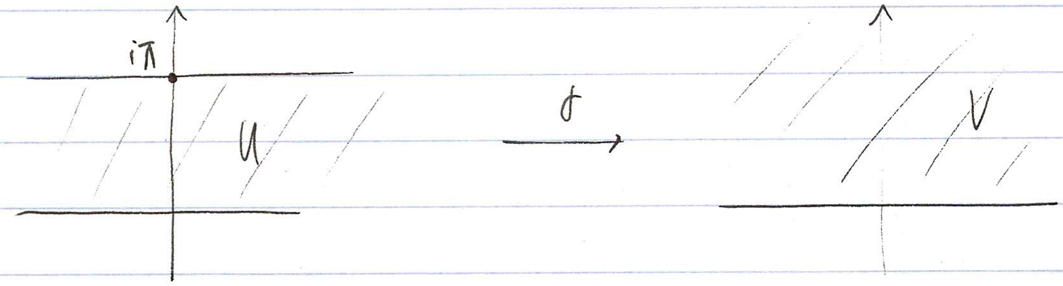


$f(z) = z^2$

(works because $f(re^{i\theta}) = r^2 e^{i2\theta} = se^{i\phi}$, $s = r^2$, $\phi = 2\theta$)

So $0 < \theta < \pi/2 \Rightarrow 0 < \phi < \pi$

c.



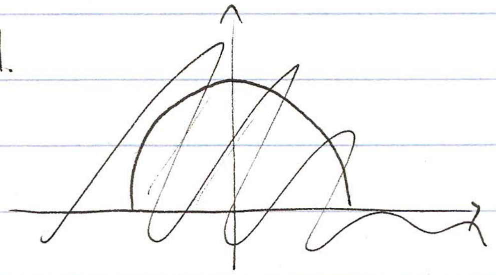
$f(z) = e^z$

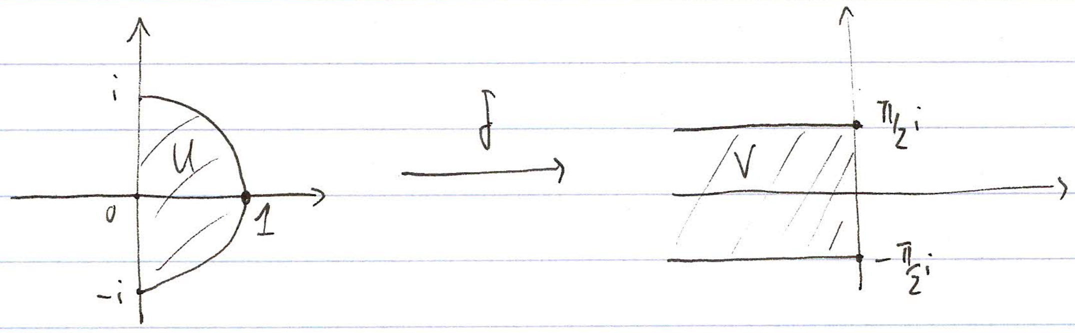
— works because $f(x+iy) = e^{x+iy} = e^x \cdot e^{iy} = se^{i\phi}$,

$s = e^x$, $\phi = y$.

So $0 < y < \pi \Rightarrow 0 < \phi < \pi$..

d.





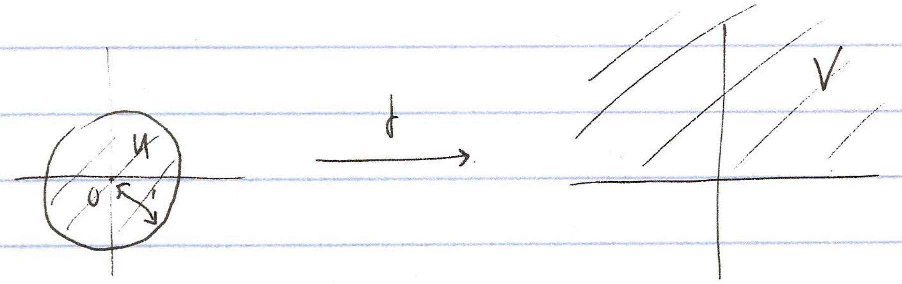
$f(z) = \text{Log}\, z$: $\text{Log}(re^{i\theta}) = \log r + i\theta$ for $-\pi < \theta \leq \pi$

$= u + iv$

So $0<r<1 \implies u=\log r$ satisfies $-\infty < u < 0$

$-\pi/2 < \theta < \pi/2 \implies -\pi/2 < v < \pi/2$.

e.



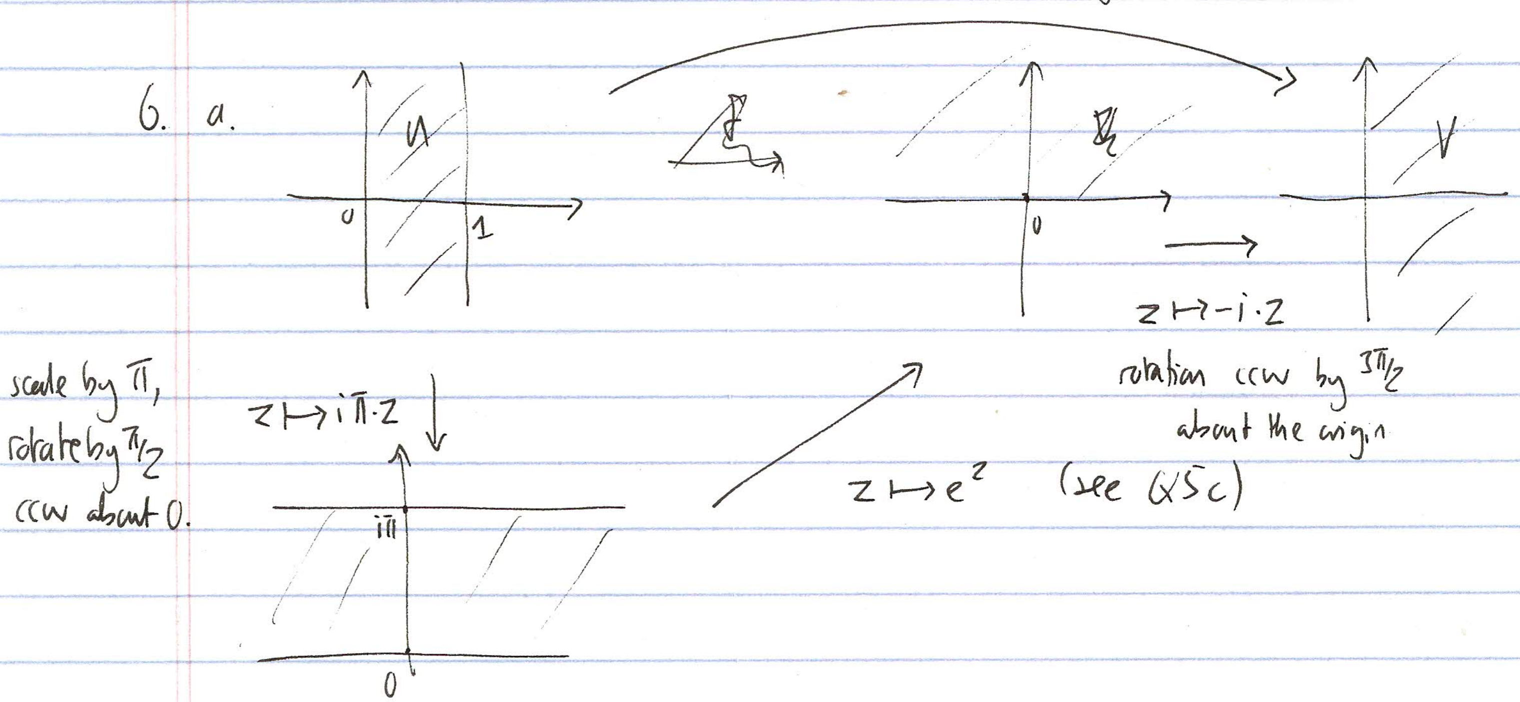
$$f(z) = -i\left(\frac{z-1}{z+1}\right) \qquad \text{LFT.}$$

(this was explained in class)

<u>Note</u>: There are other possible functions f as well.

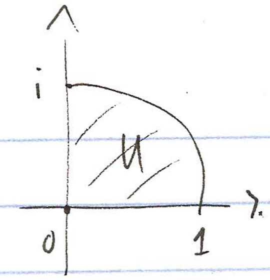
e.g. $f(z) = -i\cdot\left(\dfrac{z-i}{z+i}\right)$

6. a.

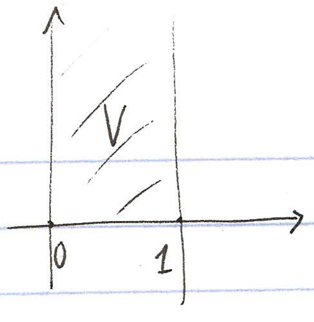


$\therefore \quad f(z) = -i\cdot e^{i\pi z}$

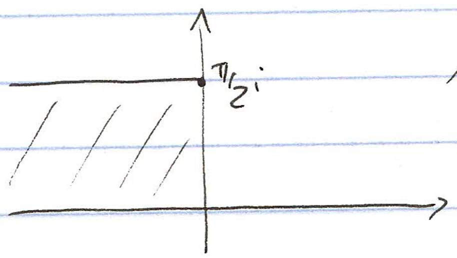
b.



$\xrightarrow{f}$



$z \mapsto \text{Log}\, z$ $\downarrow$



$z \mapsto \frac{2}{\pi} \cdot (-i) \cdot z$

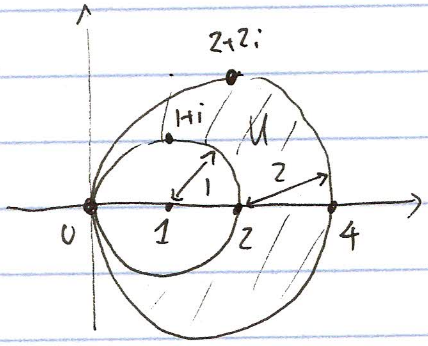
scale by $2/\pi$

rotate by $3\pi/2$ ccw.

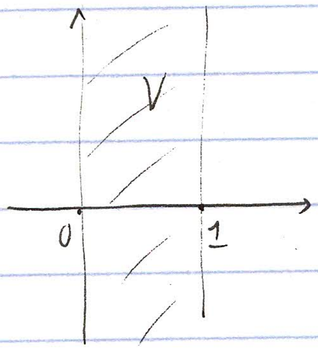
(compare Q5d)

$\therefore \quad f(z) = \frac{-2i}{\pi} \cdot \text{Log}\, z.$

c.



$\xrightarrow{f}$



$z \mapsto \frac{1}{z}$ $\downarrow$

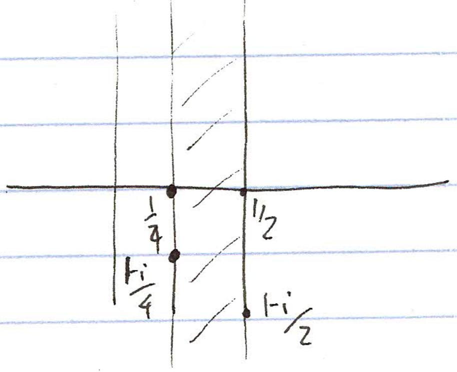
$0 \mapsto \infty$

$2 \mapsto 1/2$

$4 \mapsto 1/4$

$1+i \mapsto \frac{1-i}{2}$

$2+2i \mapsto \frac{2-2i}{8} = \frac{1-i}{4}$



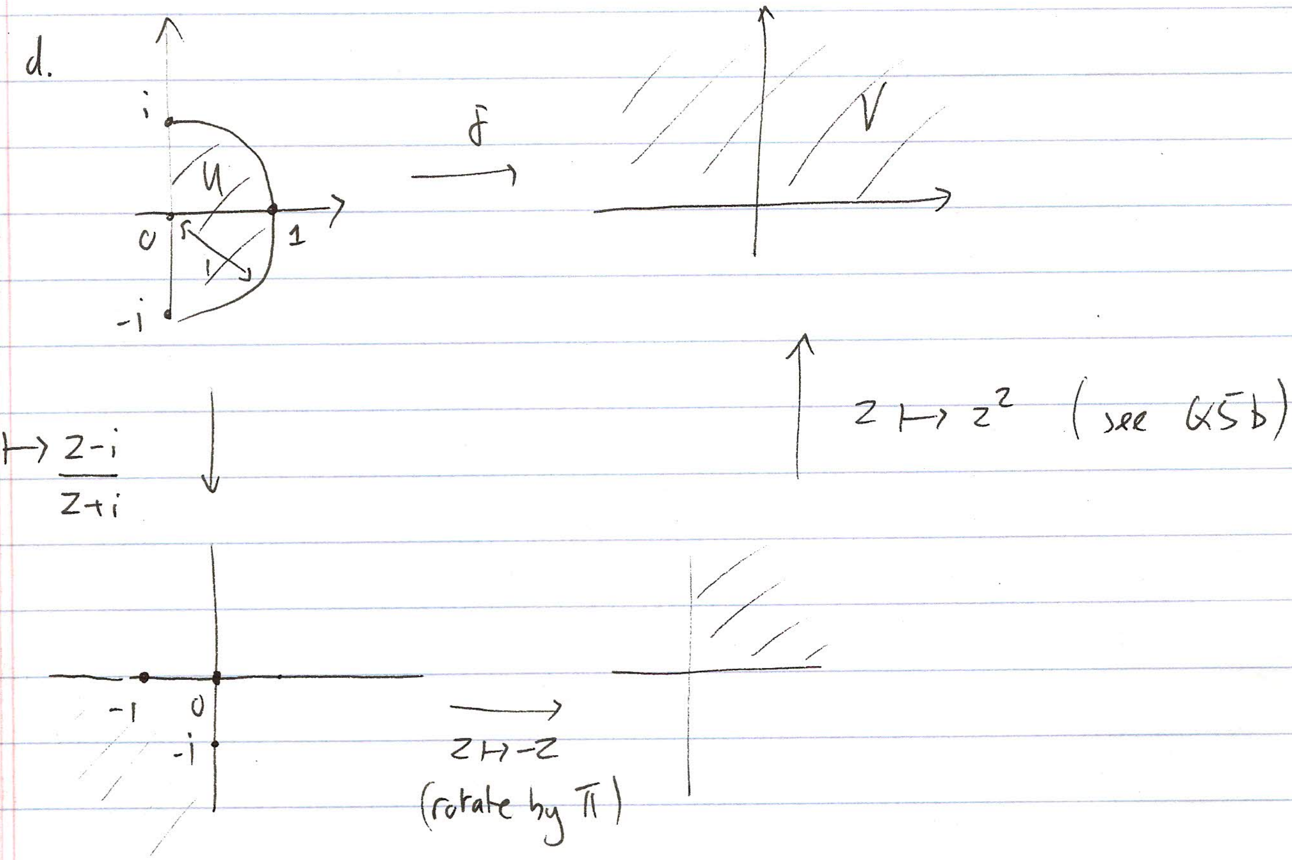
$z \mapsto 4 \cdot \left(z - \frac{1}{4}\right) = 4z - 1$

translation by $-1/4$

followed by scaling by 4.

$$\therefore \quad f(z) = 4 \cdot (1/z) - 1 = \frac{-z+4}{z}$$

d.



$i \mapsto 0$

$-i \mapsto \infty$

$0 \mapsto -1$

$1 \mapsto \frac{1-i}{1+i} = \frac{(1-i)^2}{2}$

$= -i$

$$\therefore \quad f(z) = \left(-\left(\frac{z-i}{z+i}\right)\right)^2 = \left(\frac{z-i}{z+i}\right)^2 .$$