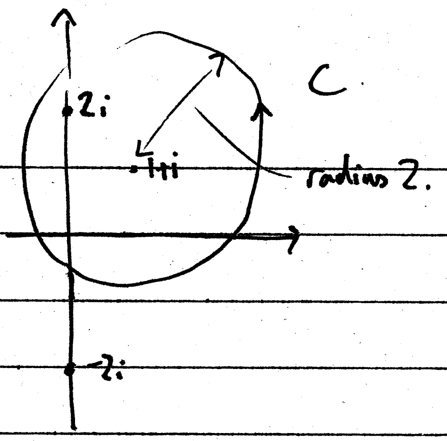
Wednesday 11/4/15

MATH 421 HW6 Solutions.

1.



$$\frac{z}{z^2+4} = \frac{z}{(z-2i)(z+2i)} = \frac{A}{z-2i} + \frac{B}{z+2i}, \quad A, B \in \mathbb{C}.$$

$$z = A(z+2i) + B(z-2i)$$

$$1\cdot z + 0 = (A+B)z + (A-B)\cdot 2i$$

$\Rightarrow$ $A+B=1$, $(A-B)\cdot 2i = 0$

$\Rightarrow$ $A+B=1$, $A-B=0$

$\Rightarrow$ $A = B = 1/2$.

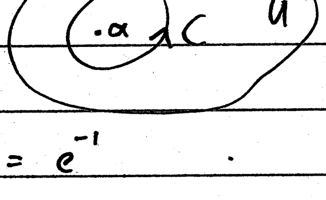
$$\int_C \frac{z}{z^2+4}\,dz = \frac{1}{2}\int_C \frac{1}{z-2i} + \frac{1}{z+2i}\,dz$$

$$= \frac{1}{2}(2\pi i + 0) = \pi i$$

$\uparrow$ $2i$ inside C, $-2i$ outside C.

$$\left(\begin{aligned} |2i - (1+i)| &= |-1+i| = \sqrt{2} < 2 \\ |-2i - (1+i)| &= |-1-3i| = \sqrt{10} > 2 \end{aligned} \right)$$

2. C.I.F. $f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}\,dz$, $f : U \xrightarrow{\mathbb{C}} \mathbb{C}$ cx. diff.



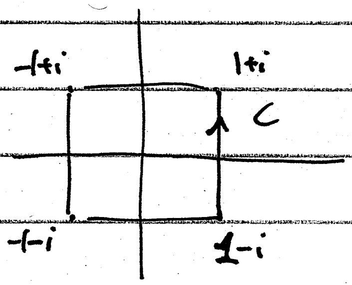
$f(z) = e^{iz}$, $\alpha = i$

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{e^{iz}}{z-i} = f(i) = e^{i^2} = e^{-1}.$$

(note i lies inside C).

$$\therefore \int_C \frac{e^{iz}}{z-i} = 2\pi i \cdot e^{-1} = \frac{2\pi i}{e}$$

3.



$$\frac{\cos z}{z^3+9z} = \frac{\cos z}{z\cdot(z^2+9)} = \frac{\cos z}{z\cdot(z-3i)(z+3i)}$$

0 is inside C, $\pm 3i$ are outside C.

$$\therefore \frac{\cos z}{z^3+9z} = \left(\frac{\cos z}{z^2+9}\right)\Big/ z = f(z)/z$$

where $f : U \longrightarrow \mathbb{C}$ cx diff'ble, $U = \mathbb{C} \setminus \{3i, -3i\}$

$C \subset U$ and (inside of C) $\subset U$

$$\text{CIF} \Rightarrow \frac{1}{2\pi i}\int_C \frac{\cos z}{z^3+9z}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz$$

$$= f(0) = \frac{\cos(0)}{0^2+9} = \frac{1}{9}$$

$$\Rightarrow \int_C \frac{\cos z}{z^3+9z}\,dz = 2\pi i/9.$$

4 $$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{it})\,dt \qquad \text{GMVT.}$$

$f(z) = z^n$, $\alpha = 0$.

LHS $= f(0) = 0^n = 0$.

$$\text{RHS} = \frac{1}{2\pi}\int_0^{2\pi} (re^{it})^n\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} r^n \cdot e^{int}\,dt$$

$$= \frac{r^n}{2\pi}\left(\int_0^{2\pi} \cos nt\,dt + i\int_0^{2\pi} \sin nt\,dt\right)$$

$$= \frac{r^n}{2\pi}\left(\left[\frac{1}{n}\sin nt \right]_0^{2\pi} + i \left[\frac{-1}{n}\cos nt \right]_0^{2\pi} \right)$$

$$= \frac{r^n}{2\pi} \cdot (0 + i \cdot 0) = 0.$$

5. $f = u + iv$

1. $f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

2. CR : a. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, b. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\therefore f' = 0 \overset{1.}{\iff} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \overset{2a.}{\iff} \frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = 0$$

$$\overset{2b}{\iff} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

So f, u, & v have same critical points.

6. $f : \mathbb{C} \to \mathbb{C}$

$f(z) = (1+i) \cdot z^2$

a.
$$\begin{aligned} f(x+iy) &= (1+i) \cdot (x+iy)^2 \\ &= (1+i) \cdot ((x^2-y^2) + i \cdot 2xy) \\ &= (x^2-y^2-2xy) + i(x^2-y^2+2xy) \\ &= \quad u \quad + \quad iv \end{aligned}$$

b. $u = x^2 - y^2 - 2xy$

$\frac{\partial u}{\partial x} = 2x - 2y$, $\frac{\partial u}{\partial y} = -2y - 2x$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \iff x = y = 0$$

Or, use CR : $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \iff f' = 0 \iff (1+i) \cdot 2z = 0$

$\iff z = 0.$

c. $\frac{\partial^2 u}{\partial x^2} = 2$, $\frac{\partial^2 u}{\partial y^2} = -2$

$\frac{\partial^2 u}{\partial x \partial y} = -2$

$\therefore \; D(x,y) = \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 2\cdot(-2) - (-2)^2 = -8 < 0$

$\Rightarrow$ saddle point.

7. $f: \mathbb{C} \to \mathbb{C}$ $f(z) = \cos z$.

a) $f(z) = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$
$= u + iv$

b) $u = \cos x \cosh y$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Leftrightarrow -\sin x \cosh y = \cos x \sinh y = 0$

Now $\cosh y \geq 1$ for all $y \in \mathbb{R}$,
in particular $\cosh y \neq 0$, so $-\sin x \cdot \cosh y = 0 \Leftrightarrow \sin x = 0$
$\Leftrightarrow x = k \cdot \pi$, k an integer

Then $\cos(k\pi) = (-1)^k \neq 0$, so $\cos x \sinh y = 0 \Leftrightarrow \sinh y = 0$
$\Leftrightarrow y = 0$.

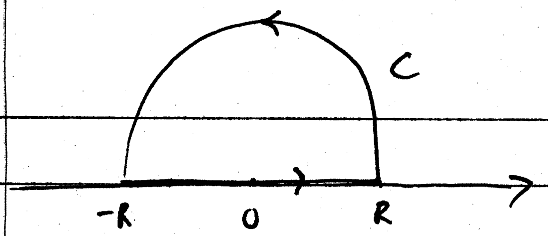
$\therefore$ critical points are $x = k \cdot \pi$, $y = 0$.

(Or use CS: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Leftrightarrow f' = 0 \Leftrightarrow -\sin z = 0$
$\Leftrightarrow z = k\pi$, k integer)
└ explained in class.

c) $\frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$, $\frac{\partial^2 u}{\partial y^2} = \cos x \cosh y$, $\frac{\partial^2 u}{\partial x \partial y} = -\sin x \sinh y$

$\therefore \; D(x,y) = -(\cos x \cosh y)^2 - (\sin x \sinh y)^2 = -1 < 0$ for $x = k\pi$, $y = 0$,
k integer.

8.



Let C be semicircular contour w/ center 0 & radius R as shown; oriented ccw.

$$\int_C \frac{1}{z^2-2z+5}\,dz = \int_{-R}^{R} \frac{1}{x^2-2x+5}\,dx + \int_{C_2} \frac{1}{z^2-2z+5}\,dz \quad \dagger.$$

where $C_1 = [-R, R]$ is the straight part of C & C_2 is the curved part.

First, observe $\lim\limits_{R\to\infty} \int_{C_2} \frac{1}{z^2-2z+5}\,dz = 0$:-

$$\left| \int_{C_2} \frac{1}{z^2-2z+5}\,dz \right| \leq \text{length}(C_2) \cdot \frac{1}{R^2-2R-5} \qquad \text{for } R \text{ sufficiently large}$$

$$= \frac{\pi R}{R^2-2R-5}$$

$$\left(\text{here } \left| \frac{1}{z^2-2z+5} \right| \leq \frac{1}{\big||z^2| - |2z-5|\big|} \leq \frac{1}{\big||z^2| - |2z| - |5|\big|} = \frac{1}{|R^2-2R-5|} = \frac{1}{R^2-2R-5} \text{ for } |z| = R \text{ \& } R \text{ suff. large.} \right)$$

$$\text{And } \lim_{R\to\infty} \frac{\pi R}{R^2-2R-5} \underset{\substack{\uparrow \\ \text{divide} \\ \text{numerator \& denominator} \\ \text{by } R^2}}{=} \lim_{R\to\infty} \frac{\pi/R}{1 - 2/R - 5/R^2} \underset{\substack{\uparrow \\ \text{"limit laws"}}}{=} \frac{0}{1} = 0.$$

$$\therefore \int_{C_2} \frac{1}{z^2-2z+5}\,dz \to 0 \text{ as } R \to \infty.$$

Now $\lim_{R\to\infty}$ of $\dagger$ gives

$$\int_{-\infty}^{\infty} \frac{1}{x^2-2x+5}\,dx = \lim_{R\to\infty} \int_C \frac{1}{z^2-2z+5}\,dz. \qquad \ddagger$$

Finally, $\dfrac{1}{z^2-2z+5} = \dfrac{1}{(z-(1+2i))(z-(1-2i))} = \dfrac{A}{z-(1+2i)} + \dfrac{B}{z-(1-2i)}$,

(Solving $z^2-2z+5=0$ using quadratic formula to find roots $1\pm 2i$)

some $A, B \in \mathbb{C}$

$$1 = A\cdot(z-(1\,\text{[illegible]}\,2i)) + B\cdot(z-(1+2i))$$

$$0\cdot z + 1 = (A+B)\cdot z - (A\cdot(1-2i) + B\cdot(1+2i))$$

$\Rightarrow A+B=0, \quad B=-A$

$$A\cdot(1-2i) + B\cdot(1+2i) = -1$$

$$A\cdot[(1-2i) - (1+2i)] = -1$$

$$A\cdot(-4i) = -1$$

$$A = 1/4i = -1/4\, i, \quad B = -A = 1/4\, i$$

$$\therefore \int_C \frac{1}{z^2-2z+5}\,dz = \frac{1}{4} i \int_C \frac{-1}{z-(1+2i)} + \frac{1}{z-(1-2i)}\,dz.$$

$$= \frac{1}{4} i \cdot (-2\pi i + 0)$$

1+2i inside C

1-2i outside C

(R suff. large)

$$= -2\pi/4\, i^2 = \pi/2$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{x^2-2x+5}\,dx \underset{\ddagger}{=} \frac{\pi}{2}$$

9. $f: \mathbb{C} \to \mathbb{C}$ cx diff'ble $\quad f(z)=\alpha$ when $|z|=R$

Let $\beta \in \mathbb{C}$, $|\beta| < R$. Let C be circle, center 0, radius R, oriented ccw

Then $$f(\beta) \underset{\text{(CIF}}{=} \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\beta}\,dz = \frac{1}{2\pi i}\int_C \frac{\alpha}{z-\beta}\,dz$$

$$= \frac{\alpha}{2\pi i}\int_C \frac{1}{z-\beta}\,dz \underset{\substack{\uparrow \\ \beta \text{ inside } C.}}{=} \frac{\alpha}{2\pi i}\cdot 2\pi i = \alpha.$$

i.e $f(z) = \alpha$ for $|z| < R$.

More generally, the CIF shows that a cmplx diffble function is determined inside C by its values on C.