1\. a $u = x+2y$, $v = 3x+4y$

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = 4$. $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow f = u+iv$ is NOT complex differentiable.

b $u = 5x+7y$, $v = -7x+5y$

$\frac{\partial u}{\partial x} = 5 = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = 7 = -\frac{\partial v}{\partial x}$ Cauchy Riemann eqs are satisfied.

Also $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

So f is complex differentiable.

c $u = 2x^2 - 6xy - 2y^2$, $v = 3x^2 + 4xy - 3y^2$

$\frac{\partial u}{\partial x} = 4x - 6y$, $\frac{\partial v}{\partial y} = 4x - 6y = \frac{\partial u}{\partial x}$

$\frac{\partial u}{\partial y} = -6x - 4y$, $\frac{\partial v}{\partial x} = 6x + 4y = -\frac{\partial u}{\partial y}$ CR eqs are satisfied.

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

$\therefore$ f is complex differentiable.

d. $u = e^y \cos x$. $v = e^y \sin x$

$\frac{\partial u}{\partial x} = -e^y \sin x$, $\frac{\partial v}{\partial y} = e^y \sin x$. $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow f$ NOT complex diff'ble

e $u = \log\sqrt{x^2+y^2}$ $v = \tan^{-1}(y/x)$

$f = u+iv$, $f : \{x+iy \mid x>0\} \longrightarrow \mathbb{C}$.

$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot 2x \cdot (x^2+y^2)^{-1/2} \cdot \frac{1}{(x^2+y^2)^{1/2}} = \frac{x}{x^2+y^2}$

$$\frac{\partial v}{\partial y} = \frac{1}{x} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} = \frac{\partial u}{\partial x}.$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \quad \left(\text{symmetric to calc. of } \frac{\partial u}{\partial x}\right)$$

$$\frac{\partial v}{\partial x} = \left(-\frac{y}{x^2}\right) \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2} = -\frac{\partial u}{\partial y}$$

$\therefore$ CR eqns are satisfied, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ cts on $U = \{x+iy \mid x > 0\}$
$\Rightarrow f$ cx differentiable on U.

2. a) $f'(z) = (3+2i) \cdot 7z^6 + 4i \cdot 2z$
$= (21+14i) \cdot z^6 + 8i \cdot z$

b) $f'(z) = \dfrac{1 \cdot (z^2+i) - (z+3) \cdot 2z}{(z^2+i)^2} = \dfrac{-z^2 - 6z + i}{(z^2+i)^2}$

c) $f'(z) = (1+i) \cdot e^{(1+i)z}$

d) $f'(z) = 1 \cdot \sin(2iz) + z \cdot 2i\cos(2iz)$
$= \sin(2iz) + 2i \cdot z\cos(2iz)$

e) $f'(z) = \dfrac{2iz}{iz^2+3}$

3. $f'(z) = \dfrac{a \cdot (cz+d) - c \cdot (az+b)}{(cz+d)^2} = \dfrac{ad-bc}{(cz+d)^2}$

$0 \neq ad-bc \in \mathbb{C}$ by assumption
$\Rightarrow f'(z) \neq 0$ for all $z \in U = \mathbb{C} \setminus \{-d/c\}$.

(Note: if $ad-bc=0$ then $az+b$, $cz+d$ are proportional & $f(z)$ is constant)

4. $f(z) = z^{\alpha} = e^{\alpha \log z}$

$\Rightarrow f'(z) = \frac{\alpha}{z} \cdot e^{\alpha \log z} = \frac{\alpha}{z} \cdot z^{\alpha} = \alpha z^{\alpha - 1}$

5. a. $z : [0,1] \longrightarrow \mathbb{C}$

$z(t) = 1+i + (3+2i - (1+i)) \cdot t$

$= 1+i + (2+i) \cdot t = (1+2t) + i(1+t)$

b. $\int_C z\, dz = \int_0^1 z(t) z'(t)\, dt$

$= \int_0^1 ((1+2t) + i(1+t)) \cdot (2+i)\, dt$

$= \int_0^1 ((2+4t) - (1+t)) + i((1+2t) + (2+2t))\, dt$

$= \int_0^1 1+3t\, dt + i \int_0^1 3+4t\, dt$

$= \left[t + 3t^2/2 \right]_0^1 + i \left[3t + 2t^2 \right]_0^1$

$= 5/2 + 5i$

c. $\int_C z\, dz = \left[z^2/2 \right]_{1+i}^{3+2i} = 1/2 \left((3+2i)^2 - (1+i)^2 \right)$

$= 1/2 (5 + 12i - 2i) = 1/2 (5+10i) = 5/2 + 5i$ ✓

6 a. $z : [0, 2\pi] \longrightarrow \mathbb{C}$ $\quad z(t) = \cos t + i \sin t$

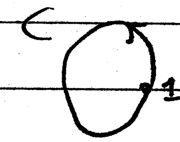
b. $\int_C z\, dz = \int_0^{2\pi} z(t) z'(t)\, dt = \int_0^{2\pi} (\cos t + i \sin t)(-\sin t + i \cos t)\, dt$

$$= \int_0^{2\pi} -2\sin t\cos t + i\left((\cos t)^2 - (\sin t)^2\right)\,dt$$

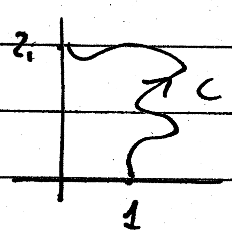
$$= \int_0^{2\pi} -\sin 2t + i\cos 2t \quad dt$$

$$= \left[\tfrac{1}{2}\cos 2t\right]_0^{2\pi} + i\left[\tfrac{1}{2}\sin 2t\right]_0^{2\pi} = 0.$$

c. $$\int_C z\,dz = \left[\,z^2/2\,\right]_1^1 = 0.$$



7.



$$\int_C (z^3 + 2iz + 3)\,dz = \left[\,z^4/4 + iz^2 + 3z\,\right]_1^{2i}$$

$$= \left((2i)^4/4 + i(2i)^2 + 6i\right) - \left(1/4 + i + 3\right)$$

$$= (4 - 4i + 6i) - (13/4 + i)$$

$$= 3/4 + i.$$

8. $$\int_C z^n\,dz = \left[\frac{z^{n+1}}{n+1}\right]_1^1 = 0.$$ $(n \neq -1 \Rightarrow n+1 \neq 0)$.