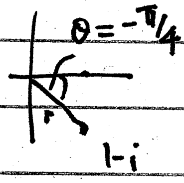
1\. a $\quad 3i = 3\cdot e^{i\pi/2} \Rightarrow \mathrm{Log}(3i) = \log 3 + i\pi/2$

b $\quad 1-i = \sqrt{2}\, e^{i(-\pi/4)} \Rightarrow \mathrm{Log}(1-i) = \log\sqrt{2} - i\pi/4$

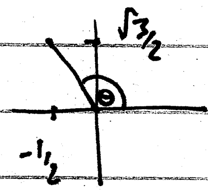


$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}.$

c $\quad -2+2\sqrt{3}\,i = 4\cdot\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right) = 4e^{i\frac{2\pi}{3}},$

$\Rightarrow \mathrm{Log}(-2+2\sqrt{3}\,i) = \log 4 + i\frac{2\pi}{3}$

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4.$



$\theta = \pi - \pi/3 = 2\pi/3$

2\. a

$$1^{1/n} = e^{1/n \log 1} \qquad 1 = 1\cdot e^{i0}$$
$$= e^{1/n(\log 1 + i(0+2\pi k))} \qquad k \text{ an integer}$$
$$= e^{2\pi i k/n} \qquad k = 0,1,2,\dots,n-1$$
$$= te^{i\psi}, \quad t = 1,\ \psi = 2\pi k/n.$$

b

$$(-i)^i = e^{i\log(-i)} = e^{i(\log 1 + i\cdot(-\pi/2 + 2\pi k))} \qquad \left(-i = 1\cdot e^{-i\pi/2}\right)$$
$$= e^{\pi/2 - 2\pi k}, \quad k \text{ an integer.}$$
$$= t\cdot e^{i\psi}, \quad t = e^{\pi/2 - 2\pi k},\ \psi = 0.$$

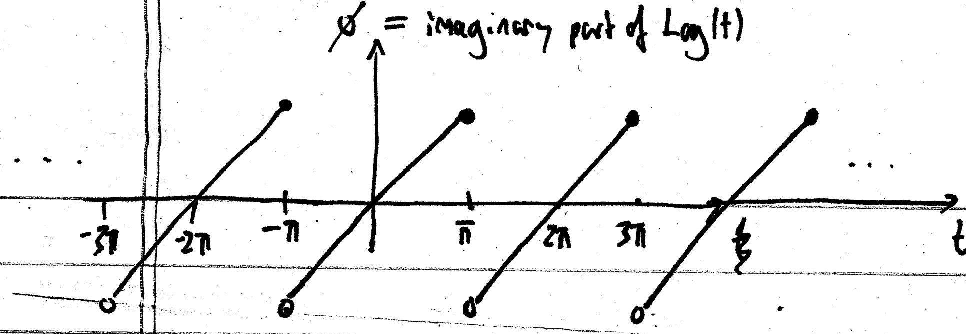
c

$$(1+i)^i = e^{i\log(1+i)} \qquad 1+i = \sqrt{2}\, e^{i\pi/4}$$
$$= e^{i(\log 2 + i(\pi/4 + 2\pi k))}$$
$$= e^{-\pi/4 - 2\pi k}\cdot e^{i\log 2} \qquad k \text{ an integer}$$
$$= te^{i\psi}, \quad t = e^{-\pi/4 - 2\pi k}, \quad \psi = \log 2$$

3\.

$$F(t) = \mathrm{Log}(\cos t + i\sin t) = \mathrm{Log}(1\cdot e^{it})$$
$$= \log 1 + i\phi = i\phi, \quad \text{where } -\pi < \phi \le \pi$$

& $\phi = t + 2\pi k$, some integer k



4. e.g. $w_1 = w_2 = e^{i3\pi/4}$.

$$\mathrm{Log}(w_1) = \mathrm{Log}(w_2) = \log 1 + i\frac{3\pi}{4} = i\frac{3\pi}{4}$$

$$w_1 w_2 = e^{i6\pi/4} = e^{i3\pi/2} = e^{-i\pi/2}$$

$$\mathrm{Log}(w_1w_2) = \log 1 + i\left(-\pi/2\right) = -i\pi/2 \neq \underbrace{\mathrm{Log}(w_1) + \mathrm{Log}(w_2)}_{=\, i3\pi/2}.$$

5. $w = se^{i\phi}$, $z = x+iy$

a $$w^z = e^{z\cdot\log w} = e^{(x+iy)(\log s + i(\phi+2\pi k))}, \quad k \text{ an integer}$$

$$= e^{x\log s - y(\phi+2\pi k) + i(y\log s + x(\phi+2\pi k))}$$

$$= te^{i\psi}, \qquad t = e^{x\log s - y(\phi+2\pi k)} \quad \dagger$$

$$\psi = y\log s + x(\phi+2\pi k) \quad \dagger\dagger$$

b If z is real, $z = x + i\cdot 0 = x$, $y=0$.

$$\overset{\dagger}{\Rightarrow} \quad t = e^{x\log s} = s^x$$

i.e. $|w^z| = |w|^x = |w|^z$.

In general, $|w^z|$ is NOT single valued, e.g.

$$|i^i| = |e^{i\log i}| = |e^{i(i(\pi/2+2\pi k))}|$$
$$= e^{-\pi/2-2\pi k}, \quad k \text{ an integer.}$$

c. If z is purely imaginary, $z = 0 + iy = iy$, $x=0$.

$\Uparrow$

$\Rightarrow \quad \psi = y\log s.$

If $|w|=1$, i.e., $s=1$, have $\psi = y\log 1 = 0$.

So w^z is real (and positive).

6\. a. $(\cosh t)^2 - (\sinh t)^2 = \left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2$

$$= \frac{1}{4}\left((e^{2t}+2+e^{-2t}) - (e^{2t}-2+e^{-2t})\right)$$

$$= \tfrac{1}{4}(4) = 1.$$

b. $x^2-y^2=1$. $\quad y = \pm\sqrt{x^2-1}$

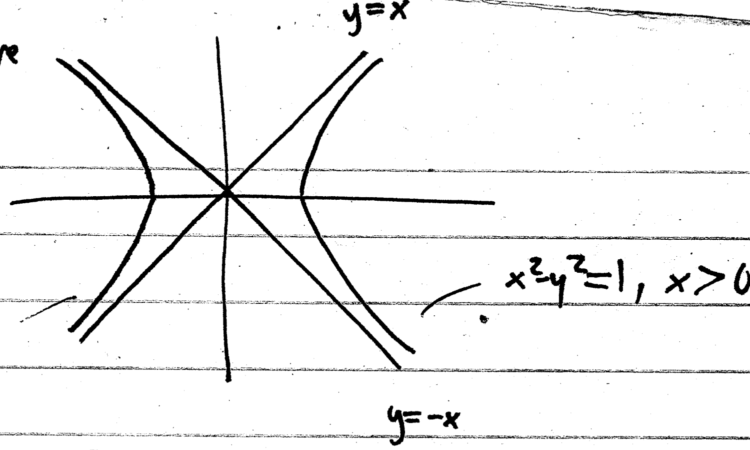
$y\in\mathbb{R} \Rightarrow x^2-1 \geq 0 \Rightarrow x\geq 1$ OR $x \leq -1$.

If $|x|$ is large, $y = \pm\sqrt{x^2-1} \approx \pm\sqrt{x^2} = \pm|x|. = \pm x.$

i.e. have asymptotes $y=x$ & $y=-x$.

Also, $y = \pm\sqrt{x^2-1} \Rightarrow |y| < \sqrt{x^2} = |x|.$

This gives picture

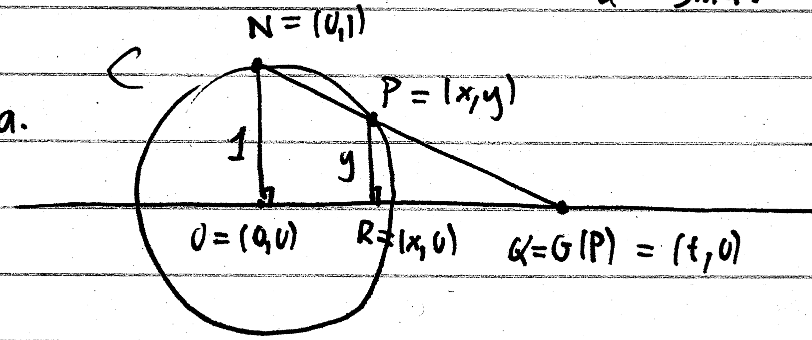


c $\quad f: \mathbb{R} \longrightarrow \mathbb{R}^2$

$f(t) = (\cosh t, \sinh t)$

gives a parametrization of the piece of the hyperbola with $x>0$.
(using part a & the facts $\cosh t > 0$ for all t
& $\sinh t \to \pm\infty$ as $t \to \pm\infty$)

7. a.



ΔNOG is similar to ΔPRG

$$\Rightarrow \frac{OG}{ON} = \frac{RG}{RP}, \quad \frac{t}{1} = \frac{t-x}{y}$$

$$\Rightarrow y\cdot t = t-x, \quad x = t\cdot(1-y), \quad t = \frac{x}{1-y}$$

$\leadsto \quad G : C\setminus\{N\} \longrightarrow \mathbb{R}$

$$G(x,y) = t = \frac{x}{1-y}$$

b. 1. $t = x/1-y$

2. $x^2+y^2=1$

1. $\Rightarrow$ $y = 1 - x/t$. Substitute in 2. $x^2 + (1-x/t)^2 = 1$.

$x^2\cdot(1+1/t^2) - 2/t\cdot x = 0$.

$$x^2 \cdot (t^2+1) - 2t \cdot x = 0$$

$$x\left(x \cdot (t^2+1) - 2t\right) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{OR} \quad x = \frac{2t}{t^2+1}$$

$$\Rightarrow \quad y = 1 - x/t = 1 \quad \text{OR} \quad y = 1 - \frac{2}{t^2+1} = \frac{t^2-1}{t^2+1}.$$

$$\therefore \quad G^{-1}(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right).$$