1.
$$z^3 = 27i$$
$$z = r(\cos\theta + i\sin\theta)$$
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta) = 27i = 27(\cos(\pi/2) + i\sin(\pi/2))$$

$$\Rightarrow \quad r^3 = 27, \quad 3\theta = \pi/2 + 2\pi k, \quad k \text{ an integer}$$

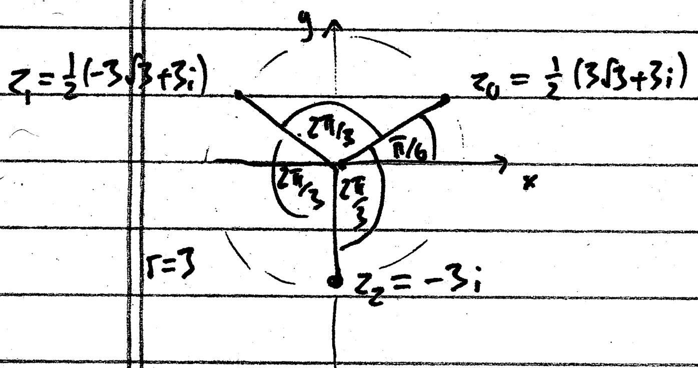
$$\Rightarrow \quad r = 3, \quad \theta = \pi/6 + 2\pi k/3, \quad k = 0,1,2$$

$$\Rightarrow \quad z_k = 3\left(\cos\left(\pi/6 + 2\pi k/3\right) + i\sin\left(\pi/6 + 2\pi k/3\right)\right), \quad k = 0,1,2.$$

$$z_0 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{1}{2}(3\sqrt{3} + 3i)$$

$$z_1 = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 3\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{1}{2}(-3\sqrt{3} + 3i)$$

$$z_2 = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 3(0 - i) = -3i$$



2.
$$z^4 = (-8 + 8\sqrt{3}i)$$
$$z = r(\cos\theta + i\sin\theta)$$
$$z^4 = r^4(\cos 4\theta + i\sin 4\theta) = -8 + 8\sqrt{3}i = s(\cos\phi + i\sin\phi)$$

$$s = \sqrt{(-8)^2 + (8\sqrt{3})^2} = \sqrt{8^2(1 + \sqrt{3}^2)} = 8\cdot\sqrt{4} = 16.$$

$$\cos\phi = \frac{-8}{16}, \quad \sin\phi = \frac{8\sqrt{3}}{16} \Rightarrow \cos\phi = \frac{-1}{2}, \ \sin\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = 2\pi/3.$$

$4\theta = \phi + 2\pi k = 2\pi/3 + 2\pi k$, k an integer

& $r^4 = s = 16$

$\Rightarrow$ $r = 2$

$\theta = 2\pi/12 + 2\pi k/4$, $k = 0,1,2,3$

$= \pi/6 + k\cdot\pi/2$, $k = 0,1,2,3$

$\Rightarrow$

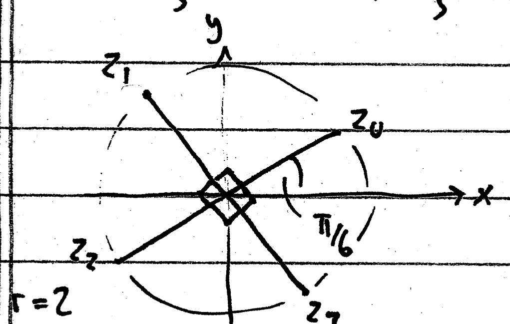
$z_k = 2\cdot\left(\cos\left(\pi/6 + k\cdot\pi/2\right) + i\sin\left(\pi/6 + k\cdot\pi/2\right)\right)$ $k = 0,1,2,3$

$z_0 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\sqrt{3}/2 + i\frac{1}{2}\right) = \sqrt{3} + i$

$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-1/2 + i\sqrt{3}/2\right) = -1 + \sqrt{3}i$

$z_2 = 2\left(\cos 7\pi/6 + i\sin 7\pi/6\right) = 2\left(-\sqrt{3}/2 - i\cdot 1/2\right) = -(\sqrt{3}+i) = -z_0$

$z_3 = 2\left(\cos 5\pi/3 + i\sin 5\pi/3\right) = 2\left(1/2 - i\sqrt{3}/2\right) = 1 - \sqrt{3}i = -z_1$



Remark: If z_0 is one solution of $z^n = c$, then the other solutions are $\rho\cdot z, \rho^2\cdot z, \dots, \rho^{n-1}\cdot z$, where $\rho = \left(\cos 2\pi/n + i\sin 2\pi/n\right)$ (why?)

In our example, $n=4$, so $\rho = i$, and

$z_1 = iz_0$, $z_2 = i^2 z_0 = -z_0$,

$z_3 = i^3 z_0 = -iz_0$

3\. Recall how the quadratic formula is derived:-

$az^2 + bz + c = 0$

"Complete the square" $0 = a\cdot\left(z^2 + \frac{b}{a}z + \frac{c}{a}\right) = a\cdot\left(\left(z + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$

$\overset{a\neq 0}{\Rightarrow}$ $\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

$\Rightarrow$ $z + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow$ $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This is still valid for $a, b, c \in \mathbb{C}$.

$$z^2 + (2+4i)z + (-3+2i) = 0. \qquad \dagger$$

$$\Rightarrow \quad z = \frac{-(2+4i) \pm \sqrt{(2+4i)^2 - 4(-3+2i)}}{2}$$

$$= -(1+2i) \pm \sqrt{(1+2i)^2 - (-3+2i)}$$

$$= -(1+2i) \pm \sqrt{-3+4i - (-3+2i)}$$

$$= -(1+2i) \pm \sqrt{2i}$$

There are two complex square roots $\pm w$ of $\sqrt{2i}$

Write $w = r(\cos\theta + i\sin\theta)$. We may assume $0 \le \theta < \pi$

Then $w^2 = r^2(\cos 2\theta + i\sin 2\theta) = 2i = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$$\Rightarrow r^2 = 2, \quad 2\theta = \pi/2, \quad \cancel{\theta = \pi/4}$$

$$\Rightarrow r = \sqrt{2}, \quad \theta = \pi/4$$

$$\Rightarrow w = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right) = 1+i.$$

So $z = -(1+2i) \pm (1+i)$

$= -i, -2-3i$ are the solutions of $\dagger$.

4. a) $z = a+bi = r(\cos\theta + i\sin\theta)$

$$\Rightarrow \sqrt{z} = \pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$= \pm\sqrt{\sqrt{a^2+b^2}}\left(\sqrt{\frac{\cos\theta+1}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right)$$

$$= \pm\sqrt[4]{a^2+b^2}\left(\sqrt{\frac{1}{2}\left(\frac{a}{\sqrt{a^2+b^2}}+1\right)} + i\sqrt{\frac{1}{2}\left(1-\frac{a}{\sqrt{a^2+b^2}}\right)}\right)$$

$$= \pm\left(\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\right)$$

$$= \pm\left(\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\right)$$

b) Check

$$\left(\sqrt{\tfrac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} + i\sqrt{\tfrac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\right)^2$$

$$= \left(\tfrac{1}{2}\left(\sqrt{a^2+b^2}+a\right) - \tfrac{1}{2}\left(\sqrt{a^2+b^2}-a\right)\right) + 2i\sqrt{\tfrac{1}{4}\left(\sqrt{a^2+b^2}+a\right)\left(\sqrt{a^2+b^2}-a\right)}$$

$i^2=-1$

$$= a + i\sqrt{\left(\sqrt{a^2+b^2}\right)^2 - a^2} = a + i\sqrt{b^2} = a+bi$$

using $b \geq 0$.

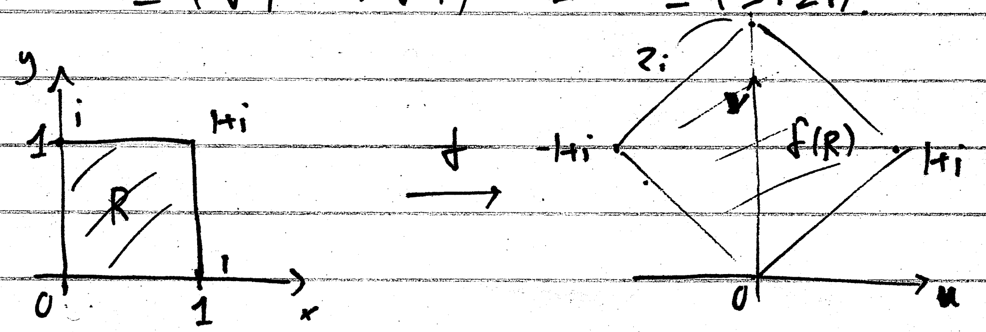
c) $z = 5+12i$

$r = \sqrt{a^2+b^2} = \sqrt{5^2+12^2} = 13.$

$$\sqrt{z} = \pm\left(\sqrt{\tfrac{1}{2}(13+5)} + i\sqrt{\tfrac{1}{2}(13-5)}\right)$$

$$= \pm\left(\sqrt{9} + i\sqrt{4}\right) = \pm(3+2i)$$

5. a)

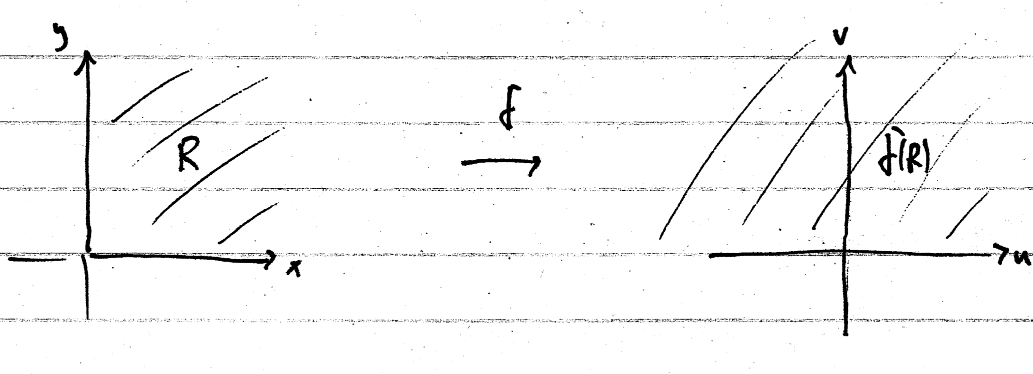


$f(z) = (1+i)\cdot z$

$(1+i) = r(\cos\theta + i\sin\theta) = \sqrt{2}\,(\cos\pi/4 + i\sin\pi/4)$

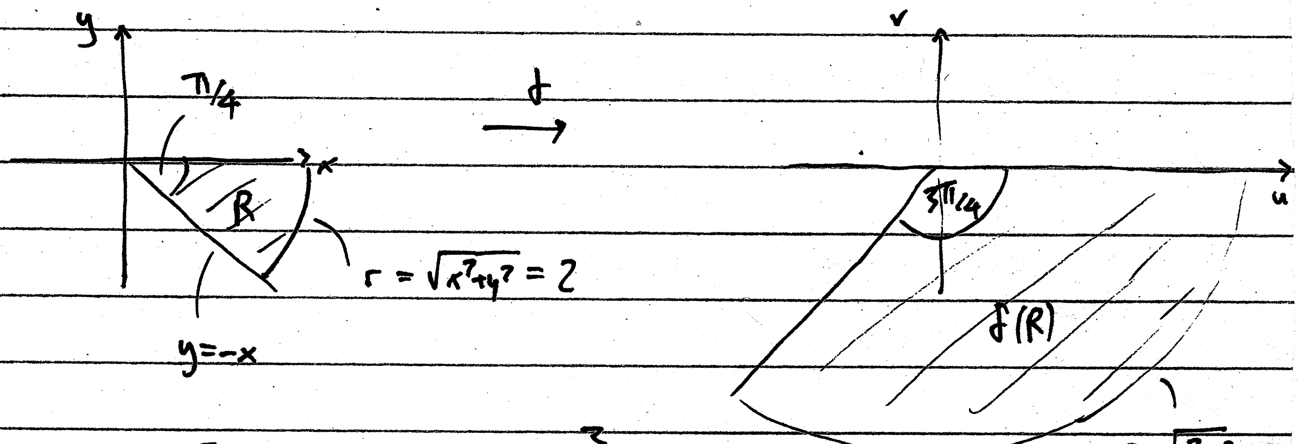
$\therefore$ f is rotation by $\pi/4$ ccw & scaling by factor $\sqrt{2}$.

b)



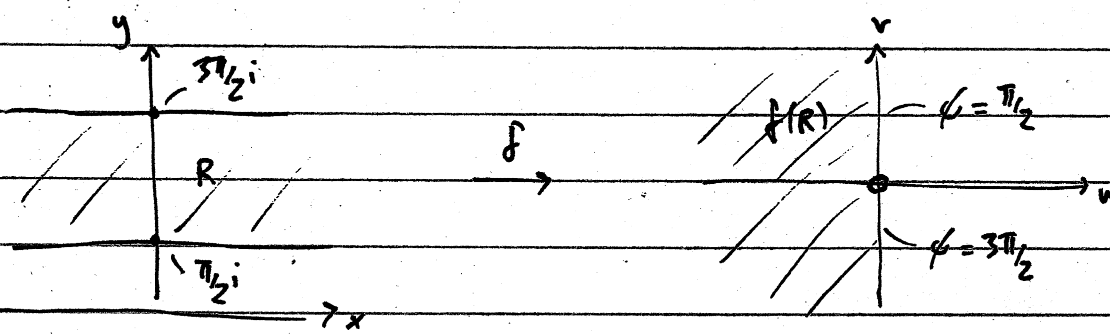
$$f(z) = z^2 = (r(\cos\theta + i\sin\theta))^2 = r^2(\cos 2\theta + i\sin 2\theta).$$

c)



$$f(z) = z^3 = (r(\cos\theta + i\sin\theta))^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

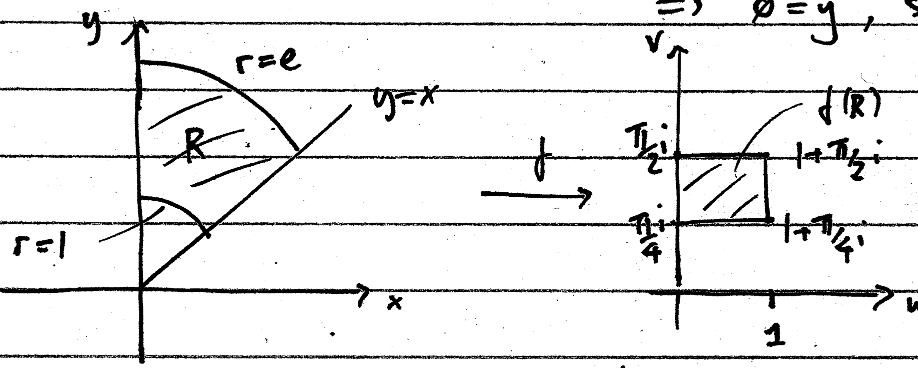
d)



$$f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y) = s(\cos\phi + i\sin\phi) = u+iv.$$

$$\Rightarrow \phi = y, \quad s = e^x.$$

e)



$$f(z) = \mathrm{Log}(z) = w \quad \Rightarrow z = e^w = e^{u+iv} = e^u(\cos v + i\sin v)$$

$$\| $$

$$r(\cos\theta + i\sin\theta)$$

$$\Rightarrow r = e^u, \quad v = \theta + 2\pi k, \ k \text{ an integer}, \quad -\pi < v \le \pi.$$

$$\Rightarrow u = \log r, \quad v = \theta \quad (-\pi < \theta \le \pi)$$

6. $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$\sin(z) = 0 \iff e^{iz} = e^{-iz}$$
$$\iff e^{2iz} = 1$$
$$\iff 2iz = (2\pi i)\cdot k, \text{ some integer } k$$
$$\iff z = \pi k \quad, \text{ some integer } k.$$

Note: All solutions are real (the usual solutions of $\sin(x) = 0$ for $x \in \mathbb{R}$)

7. $$\cos(z+\pi) = \frac{e^{i(z+\pi)} + e^{-i(z+\pi)}}{2} \qquad \left(\text{using } \cos z = \frac{e^{iz}+e^{-iz}}{2}\right)$$
$$= \frac{e^{iz}\cdot e^{i\pi} + e^{-iz}\cdot e^{-i\pi}}{2}$$
$$= \frac{-e^{iz} - e^{-iz}}{2} = -\cos z.$$

(using $e^{iy} = \cos(y) + i\sin y$
$\Rightarrow e^{i\pi} = \cos(\pi) + i\sin\pi = -1$)

8. a $$f(z) = z + \frac{1}{z} = \alpha$$

$$\iff z^2 + 1 = \alpha z$$
$$\iff z^2 - \alpha z + 1 = 0$$
$$\iff z = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

$\therefore$ There are two solutions if $\alpha^2 - 4 \neq 0$
and one solution if $\alpha^2 - 4 = 0$
And $\alpha^2 - 4 = 0 \iff \alpha = \pm 2$.
So, have two solutions if $\alpha \neq \pm 2$, & one solution if $\alpha = \pm 2$.
In particular the range of f is $\mathbb{C}$, i.e., f is onto.

b. $g : \mathbb{C} \to \mathbb{C}$

$g(z) = \cos z.$

We have $g(z) = \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Note that $e^{-w} = 1/e^{w}$ (because $e^{-w} \cdot e^{w} = e^{-w+w} = e^{0} = 1$)

So $g(z) = \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right)$,

Observe $g(z) = a(f(b(c(z))))$

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "c"] \arrow[rrrr, bend left, "g"] & \mathbb{C} \arrow[r, "b"] & \mathbb{C}\setminus\{0\} \arrow[r, "f"] & \mathbb{C} \arrow[r, "a"] & \mathbb{C}
\end{tikzcd}
$$

$z \longmapsto iz$

$w \longmapsto e^{w}$

$v \longmapsto f(v) = v + \frac{1}{v}$

$u \longmapsto \frac{1}{2}u$

i.e g is a composite function as shown,

where $c(z) = iz$, $c : \mathbb{C} \to \mathbb{C}$

$b(w) = e^{w}$, $b : \mathbb{C} \to \mathbb{C}\setminus\{0\}$

$f(v) = v + \frac{1}{v}$, $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$

$a(u) = \frac{1}{2}u$, $a : \mathbb{C} \to \mathbb{C}.$

Since each of the functions c, b, f, a is onto, the composite $g(z) = \cos z$ is also onto, i.e., the range of g equals $\mathbb{C}$.