Math 300.3 Midterm 2 review

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April 5, 2017

The syllabus for the midterm is the following sections of Sundstrom:
3.1, 3.2, 3.3, 3.4, 3.5, 4.1, 4.2, 4.3, 8.1, 8.2, and 8.3

Midterm 2 is Tuesday 4/11/17, 7:00PM–8:30PM, in LGRT 206.

The review session for midterm 2 is Monday 4/10/17, 7:00PM–8:30PM, in LGRT 204.

Justify your answers carefully.

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x| + |x-1|$ for all $x \in \mathbb{R}$. Give a proof using cases that $f(x) \geqslant 1$ for all $x \in \mathbb{R}$.

(2) Find all solutions $x \in \mathbb{Z}$ of the following congruences and express your answer in congruence notation.

  (a) $x^2 + x \equiv 0 \bmod 6$.
  (b) $x^3 + 1 \equiv 0 \bmod 7$.

(3) (a) Give a proof using cases that the congruence $x^2 \equiv 2 \bmod 5$ has no solutions $x \in \mathbb{Z}$.
  (b) Using part (a) or otherwise, give a proof by contradiction of the following statement: The equation $x^2 - 5y^2 = 7$ has no solutions $x, y \in \mathbb{Z}$.

(4) For each $n \in \mathbb{N}$ define

$$a_n = 1 - 3 + 5 - 7 + \cdots + (-1)^{n-1}(2n-1) = \sum_{k=1}^{n} (-1)^{k-1}(2k-1).$$

Guess a formula for a_n and prove your formula is correct using induction.

(5) Prove the following statement by induction: For all $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

(6) For $n \in \mathbb{N}$, we say a function $f\colon \mathbb{R} \to \mathbb{R}$ is a *polynomial of degree* n if there are coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$ such that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad \text{for all } x \in \mathbb{R}.$$

If $f(x)$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, then we can write

$$f(x) = (x - \alpha)g(x) + r$$

where $g(x)$ is a polynomial of degree $n-1$ and $r \in \mathbb{R}$. (This is a special case of "division with remainder" for polynomials.)

(a) With notation as above, show that $f(\alpha) = r$. In particular, if $f(\alpha) = 0$ then $f(x) = (x - \alpha)g(x)$.

(b) Prove the following statement by induction using part (a): If f is a polynomial of degree n then there are at most n real solutions of the equation $f(x) = 0$.

(7) Prove the following statement by induction: For all $n \in \mathbb{N}$ such that $n \geqslant 5$, $n! \geqslant 3^{n-1}$.

(8) Define a sequence $a_1, a_2, a_3, \ldots$ recursively by $a_1 = 1$, $a_2 = 7$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geqslant 3$. Prove the following statement by strong induction: For all $n \in \mathbb{N}$, $a_n = 3 \cdot 2^n - 5$.

(9) For which $c \in \mathbb{Z}$ does the equation $492x + 213y = c$ have a solution $x, y \in \mathbb{Z}$?

(10) Find all solutions $x, y \in \mathbb{Z}$ of the equation $52x + 91y = 26$.

(11) Find all solutions $x \in \mathbb{Z}$ of the congruence $42x \equiv 12 \bmod 57$ and express your answer in congruence notation.

(12) Give a proof by contradiction of the following statement: There does not exist an integer x such that $x \equiv 9 \bmod 14$ and $x \equiv 5 \bmod 21$.

(13) Give a direct proof of the following statement: For all $a, b, c \in \mathbb{N}$, if $c \mid a$ and $c \mid b$ then $c \mid \gcd(a, b)$.

[Hint: By the Euclidean algorithm, there exist integers x and y such that $ax + by = \gcd(a, b)$.]

(14) Give a direct proof of the following statement: For all prime numbers p and positive integers a, if $a < p$ then $\gcd(a, p) = 1$.

(15) For $n \in \mathbb{N}$, what are the possible values of (a) $\gcd(n, n + 7)$ and (b) $\gcd(7n + 11, 3n + 5)$? Justify your answers carefully.

(16) For $a, b \in \mathbb{N}$ we define the *least common multiple* $\operatorname{lcm}(a, b)$ of a and b to be the smallest positive integer l such that $a \mid l$ and $b \mid l$.

(a) Suppose we write a and b as products of primes:

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

and

$$b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$$

where r is a non-negative integer, $p_1, p_2, \ldots, p_r$ are distinct primes and α_i and β_i are non-negative integers for each $i = 1, 2, \ldots, r$. Determine $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ in terms of these factorizations.

[Note: We allow $\alpha_i = 0$ or $\beta_i = 0$ to account for primes which divide a but not b or vice versa.]

(b) Using part (a) or otherwise, prove that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$ for all $a, b \in \mathbb{N}$.

(17) Consider the sequence of primes listed in increasing order:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

Give a proof by contradiction of the following statement: If p and q are consecutive odd primes in the list, then $p + q$ has at least 3 prime factors (not necessarily different). For example:

$$3 + 5 = 8 = 2 \cdot 2 \cdot 2$$

$$5 + 7 = 12 = 2 \cdot 2 \cdot 3$$

$$7 + 11 = 18 = 2 \cdot 3 \cdot 3$$