Math 300.3 Midterm 1 review

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The syllabus for the midterm is the following sections of Sundstrom:
2.1, 2.2, 2.3, 2.4, 3.1, 3.2, and 3.3.

Midterm 1 is Tuesday 2/28/17, 7:00PM–8:30PM, location TBA.

The review session for midterm 1 is Monday 2/27/17, 7:00PM–8:30PM, in LGRT 202.

Justify your answers carefully.

(1) For each of the following pairs of compound statements, show using truth tables that the statements are logically equivalent.

   (a) $P \Rightarrow Q$, $(\text{NOT}\, P)\, \text{OR}\, Q$.

   (b) $(P\, \text{AND}\, Q) \Rightarrow R$, $P \Rightarrow (Q \Rightarrow R)$.

(2) Write down the negations of the following statements (simplify your answer as much as possible). Here P and Q are statements, U is a set, and $P(x)$ is a sentence involving a variable x which becomes a statement when x is replaced by an element of U.

   (a) $P\, \text{AND}\, Q$.

   (b) $P\, \text{OR}\, Q$.

   (c) $P \Rightarrow Q$.
   [Hint: Use Q1(a)]

   (d) $(\forall x \in U)(P(x))$

   (e) $(\exists x \in U)(P(x))$

(3) For each of the following false statements, (i) translate the statement into symbolic form using quantifiers, (ii) form the negation of the statement in symbolic form, (iii) translate the negation into an english sentence, and (iv) give a proof of the negation.

  (a) There is an integer n such that $n^2 + n$ is odd.
  (b) For all real numbers x, if $x^2 > 9$ then $x > 3$.
  (c) There is a real number x such that for all real numbers y we have $y \leqslant x$.
  (d) For all real numbers x there is a real number y such that $y^2 = x$.
  (e) There is a real number x such that $x \geqslant 0$ and $e^x < 1$.
  (f) For all integers n, if 8 divides $n^2 - 1$ then 8 divides $n - 1$ or 8 divides $n + 1$

(4) For each of the following statements, give a proof or a counterexample.

  (a) For all positive integers a and integers b and c, if $a \mid b$ and $a \mid c$ then $a \mid 4b - 5c$
  (b) For all positive integers a and integers b and c, if $a \mid bc$ then $a \mid b$ or $a \mid c$.
  (c) For all positive integers a and b and integers c, if $a \mid c$ and $b \mid c$ then $ab \mid c$.
  (d) For all positive integers a and b and integers c, if $a^2 \mid b$ and $b^3 \mid c$ then $a^6 \mid c$.

(5) Prove the following statement: For all positive integers n, if $n > 2$ then $n^2 - 1$ is not prime.

(6) (a) Prove the following statement: For all real numbers x,

$$x^3 + 1 = (x + 1)(x^2 - x + 1).$$

  (b) Using part (a) or otherwise, prove the following statement: For all positive integers n, if $n > 1$ then $n^3 + 1$ is not prime.

(7) Prove the following existence statements.

  (a) There is a real number x such that $x^2 + 7x + 5 = 0$.

(b) There is a real number x such that $x^3 + x = 3$.

(c) There is a real number x such that $e^x = x^2$.

(8) Consider the following conditional statement: For all real numbers x, if x^2 is irrational then x is irrational.

(a) Write down the contrapositive statement and prove it.

(b) Write down the converse statement. Is the converse true or false? (Give a proof or a counterexample.)

(9) Prove the following biconditional statements.

(a) For all real numbers x, $x^5 + 6x^3 = 0 \iff x = 0$.

(b) For all integers a, a is odd $\iff 4 \mid a^2 - 1$.

[Hint: For the reverse implication, consider the contrapositive statement.]

(10) Using a proof by contradiction or otherwise, prove the following statement: There do not exist real numbers x, y, and z such that

$$x + y + z = 1,\ x + 2y + 3z = 2,\ \text{and } 2y + 4z = 3.$$

(11) Using a proof by contradiction or otherwise, prove the following statements.

(a) $\sqrt{6}$ is irrational.

[Hint: Adapt the proof that $\sqrt{2}$ is irrational given in class, using the following results. For all integers a and b, if ab is even then a is even or b is even. For all integers a, if a^2 is even then a is even (this is the special case $a = b$ of the previous result).]

(b) $\sqrt{2} + \sqrt{3}$ is irrational.

[Hint: Compute $(\sqrt{2} + \sqrt{3})^2$ and use part (a).]