# Math 300.3 Final exam review questions

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Syllabus: Sundstrom, 2.1, 2.2, 2.3, 2.4, 3.1, 3.2, 3.3, 3.4, 3.5, 4.1, 4.2, 4.3, 5.1, 5.2, 5.3, 6.1, 6.3, 6.4, 6.5, 7.1, 7.2, 7.3, 8.1, 8.2, 8.3.

See also the class log at http://people.math.umass.edu/ hacking/300S17/classlog/

Justify your answers carefully.

(1) Let $P$ and $Q$ be statements.

(a) What is the contrapositive of $P \Rightarrow Q$? Use a truth table to show that $P \Rightarrow Q$ and its contrapositive are equivalent.

(b) What is the converse of the statement $P \Rightarrow Q$? Use a truth table to show that $P \Rightarrow Q$ and its converse are *not* equivalent.

(2) Let $A, B, C$ be sets.

(a) Define the union $A \cup B$, the intersection $A \cap B$, and the difference $A \backslash B$.

(b) Show using a truth table that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Check this result using a Venn diagram.

(c) Show using a truth table that $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$. Check this result using a Venn diagram.

(3) Translate the following statements into english sentences.

(a) $(\forall x \in \mathbb{N})(x \geqslant 1)$

(b) $(\forall x \in \mathbb{R})(x^2 \geqslant 0)$

(c) $(\exists x \in \mathbb{R})(x^2 - 6x + 7 = 0)$

(d) $(\exists x \in \mathbb{Z})(x^2 \equiv 2 \bmod 7)$
(e) $(\forall x, y \in \mathbb{R})((xy = 0) \Rightarrow ((x = 0)\,\mathrm{OR}(y = 0)))$
(f) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y > x)$
(g) $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x^3 = y)$

(4) Negate the following statements, then translate into an english sentence.

(a) $(\exists x \in \mathbb{Z})(x^2 \equiv 3 \bmod 4)$
(b) $(\forall x \in \mathbb{R})(x^2 - 4x + 2 > 0)$
(c) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x)$
(d) $(\exists b \in \mathbb{R})(\forall x \in \mathbb{R})(\log x \leqslant b)$
(e) $(\exists x, y, z \in \mathbb{N})(x^3 + y^3 = z^3)$

(5) Translate the following sentences into mathematical statements using quantifiers.

(a) $x^2 + 2x + 3$ is positive for all real numbers $x$.
(b) There is a real number $x$ such that $x^2 = 2$.
(c) For every positive integer $n$ there is a real number $a$ such that $e^x \geqslant x^n$ for $x \geqslant a$.
(d) There is a real number $b$ such that $x - x^2 \leqslant b$ for all real numbers $x$.

(6) Define a sequence of integers $a_1, a_2, a_3, \ldots$ recursively by $a_1 = 10$ and $a_{n+1} = 3a_n - 8$ for $n \in \mathbb{N}$. Prove that $a_n = 2 \cdot 3^n + 4$ for all $n \in \mathbb{N}$.

(7) Prove that

$$\sum_{r=1}^{n}(2r+1) = 3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$$

for each $n \in \mathbb{N}$.

(8) Prove that

$$\sum_{r=1}^{n} r(r+2) = 1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$$

for each $n \in \mathbb{N}$.

(9) Prove that $5^n > 4^n + 3^n + 2^n$ for all $n \in \mathbb{N}$ such that $n \geqslant 3$.

(10) Define the Fibonacci numbers $f_n$ for $n \in \mathbb{N}$ by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geqslant 2$.

(a) Write down the first few terms of the Fibonnaci sequence $f_1, f_2, f_3, \ldots$.

(b) Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n \cdot f_{n+1}$ for each $n \in \mathbb{N}$.

(11) Let $a, b \in \mathbb{N}$. What is the definition of the greatest common divisor $\gcd(a, b)$? Compute the greatest common divisor of the following pairs of integers

(a) $123, 39$.

(b) $157, 83$.

(c) $2^5 \cdot 3^7 \cdot 5^9 \cdot 11^4$, $2 \cdot 3^2 \cdot 7^{10}$.

(12) Prove that $\gcd(3n + 2, 3n + 5) = 1$ for all $n \in \mathbb{N}$.

(13) Find all solutions $x, y \in \mathbb{Z}$ of the following equations.

(a) $24x + 52y = 8$

(b) $42x + 15y = 7$

(14) Find all solutions of the following congruences.

(a) $5x \equiv 12 \bmod 17$.

(b) $x^2 + 3x + 1 \equiv 0 \bmod 5$.

(15) (a) Show that the congruence $x^3 + x + 1 \equiv 0 \bmod 4$ has no solutions.

(b) Using part (a) or otherwise show that the equation $x^3 + x = 4y^2 + 7$ has no solutions $x, y \in \mathbb{Z}$.

(16) What does it mean to say a positive integer $n > 1$ is prime?

(a) State the fundamental theorem of arithmetic.

(b) List all the positive integers $d$ such that $d \mid 108$.

(c) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factorization of a positive integer $n$. How many positive integers $d$ such that $d \mid n$ are there?

(17) Prove the following statement: For all $a, b \in \mathbb{N}$, if $a^2 \mid b^2$ then $a \mid b$.

(18) Let $A$ and $B$ be sets and $f\colon A \to B$ a function from $A$ to $B$. What does it mean to say that $f$ is injective? What does it mean to say that $f$ is surjective? For each of the following functions determine whether $f$ is injective and whether $f$ is surjective. (Justify your answers carefully.)

(a) $f\colon [0, \pi] \to \mathbb{R}$, $f(x) = 2\sin x + 5$.
(b) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x$.
(c) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + y^2$.
(d) $f\colon \mathbb{N}^3 \to \mathbb{N}$, $f(x, y, z) = 2^x \cdot 3^y \cdot 5^z$.

(19) Let $a, b \in \mathbb{N}$.

(a) Show that the function $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ defined by $f(x, y) = ax + by$ is surjective if and only if $\gcd(a, b) = 1$.
(b) Is $f$ injective? Justify your answer.

(20) Let $m \in \mathbb{N}$ and let $a \in \mathbb{N}$ be such that $\gcd(a, m) = 1$

(a) Let $A = \{0, 1, 2, \ldots, m-1\}$. Let $f\colon A \to A$ be the function defined as follows: for all $x \in A$, $f(x)$ is the remainder when $ax$ is divided by $m$. (In other words, $f(x) \in A$ is determined by $f(x) \equiv ax \bmod m$.) Prove that $f$ is injective.
(b) Deduce that $f$ is bijective.

(21) Let $A$ and $B$ be sets and $f\colon A \to B$ a function from $A$ to $B$. What condition must $f$ satisfy in order to have an inverse? In each of the following cases determine whether $f$ has an inverse and if so describe the inverse explicitly.

(a) $f\colon \mathbb{R} \to [5, \infty)$, $f(x) = 4e^x + 5$.
(b) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x + 8$.
(c) $f\colon [1, 2] \to [3, 6]$, $f(x) = x^2 - 6x + 11$.
(d) $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $f(x) = 1 + \frac{1}{x}$.
(e) $f\colon (0, \infty) \to \mathbb{R}$, $f(x) = x - \frac{1}{x}$.

(22) Let $A, B$ be sets and $f\colon A \to B$, $g\colon B \to A$ be functions.

(a) Show that if $g(f(x)) = x$ for all $x \in A$ and $f$ is surjective, then $f(g(y)) = y$ for all $y \in B$. (So $g$ is the inverse of $f$ and $f$ is bijective).

(b) Give an example of functions $f$ and $g$ such that $g(f(x)) = x$ for all $x \in A$ but $f(g(y)) \neq y$ for some $y \in B$.

(23) Let $A$ and $B$ be finite sets such that $|A| = m$ and $|B| = n$.

(a) Prove that there are $n^m$ functions from $A$ to $B$.

(b) Prove that there are

$$n!/(n-m)! = n(n-1)(n-2)\cdots(n-m+1)$$

injective functions from $A$ to $B$ if $m \leqslant n$ and none if $m > n$.

(c) How many surjective functions from $A$ to $B$ are there?

[Hint for (c): Let $S$ be the set of all functions from $A$ to $B$. Write $B = \{b_1, b_2, \ldots, b_n\}$. For each $i = 1, 2, \ldots, n$, let $S_i \subset S$ be the set of functions $f$ such that $b_i$ is *not* in the range of $f$. So the set of surjective functions equals $S \backslash (S_1 \cup \cdots \cup S_n)$. Now use the inclusion-exclusion principle to find a formula for the number of surjective functions. What happens if $m < n$?]

(24) Let $S$ be a set and $R$ a relation on $S$. What does it mean to say that $R$ is an equivalence relation? In each of the following cases, determine whether $R$ is an equivalence relation.

(a) $S = \mathbb{R}$, $aRb \iff a \leqslant b$.

(b) $S = \mathbb{R}$, $aRb \iff b = a + \pi n$ for some $n \in \mathbb{Z}$.

(c) $S = \mathbb{N}$, $aRb \iff a \mid b$.

(d) $S = \mathbb{Z}$, $aRb \iff 7 \mid a - b$.

(e) $S = \mathbb{R}$, $aRb \iff |a - b| \leqslant 2$.

(f) $S = \mathbb{R}^2$, $(a_1, a_2)R(b_1, b_2) \iff \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} < 5$.

(g) $S = \mathbb{Q}$, $aRb \iff b = 3^n a$ for some $n \in \mathbb{Z}$.

(h) $S = \mathbb{N}$, $aRb \iff ab = n^2$ for some $n \in \mathbb{N}$.

(i) $S = \mathbb{Z}^2$, $(a_1, a_2)R(b_1, b_2) \iff a_1 b_2 = a_2 b_1$.

(25) Let $S = \{1, 2, 3, 4, 5, 6\}$ and let $R$ be the relation on $S$ defined by the following table. (In row $a$ and column $b$ of the table we write $Y$ if $aRb$ and $N$ otherwise.) Determine whether $R$ is an equivalence relation and if so list the equivalence classes.

| $R$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $Y$ | $N$ | $N$ | $N$ | $Y$ | $N$ |
| 2 | $N$ | $Y$ | $Y$ | $N$ | $N$ | $Y$ |
| 3 | $N$ | $Y$ | $Y$ | $N$ | $N$ | $Y$ |
| 4 | $N$ | $N$ | $N$ | $Y$ | $N$ | $N$ |
| 5 | $Y$ | $N$ | $N$ | $N$ | $Y$ | $N$ |
| 6 | $N$ | $Y$ | $Y$ | $N$ | $N$ | $Y$ |

(26) Let $S = \mathbb{R}^2$ and let $R$ be the relation on $S$ defined by

$$(x_1, y_1)R(x_2, y_2) \iff x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

(a) Show that $R$ is an equivalence relation.

(b) Draw a picture showing the equivalence classes of $R$ in $\mathbb{R}^2$.

(27) Let $S = \mathbb{R}^2 \backslash \{(0, 0)\}$ and $R$ be the relation on $S$ defined by

$$(x_1, y_1)R(x_2, y_2) \iff (x_2, y_2) = \lambda(x_1, y_1) \text{ for some positive real number } \lambda.$$

(a) Show that $R$ is an equivalence relation.

(b) Draw a picture showing the equivalence classes of $R$ in the plane $\mathbb{R}^2$.

(c) Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the circle with center the origin and radius 1. Let $f$ be the function

$$f : C \to S/R$$

from the circle $C$ to the set $S/R$ of equivalence classes of $R$ given by $f(x, y) = [(x, y)]$ (that is, $f(x, y)$ is the equivalence class of $(x, y)$). Show that $f$ is a bijection.