Math 300.2 Midterm 2 review

Paul Hacking

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The syllabus for the midterm is the following sections of Sundstrom:

$$3.1, 3.2, 3.3, 3.4, 3.5, 4.1, 4.2, 4.3, 8.1, 8.2, 8.3, 5.1, 5.2, \text{ and } 5.3.$$

You should also know the binomial theorem, the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for binomial coefficients, and the inclusion-exclusion principle. (These were discussed in class but are not covered in the text.)

Midterm 2 is Wednesday 11/15/17, 7:00PM–8:30PM, in LGRT 143.

The review session for midterm 2 is Tuesday 11/14/17, 7:00PM–8:30PM, in LGRT 143.

Justify your answers carefully.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x| + |x-1|$ for all $x \in \mathbb{R}$. Give a proof using cases that $f(x) \geqslant 1$ for all $x \in \mathbb{R}$.

(2) Find all solutions $x \in \mathbb{Z}$ of the following congruences and express your answer in congruence notation.

  (a) $x^2 + x \equiv 0 \bmod 6$.

  (b) $x^3 + 1 \equiv 0 \bmod 7$.

(3) (a) Give a proof using cases that the congruence $x^2 \equiv 2 \bmod 5$ has no solutions $x \in \mathbb{Z}$.

  (b) Using part (a) or otherwise, give a proof by contradiction of the following statement: The equation $x^2 - 5y^2 = 7$ has no solutions $x, y \in \mathbb{Z}$.

(4) Prove the following statement by induction: For all $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$

(5) For $n \in \mathbb{N}$, we say a function $f\colon \mathbb{R} \to \mathbb{R}$ is a *polynomial of degree n* if there are coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$ such that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad \text{for all } x \in \mathbb{R}.$$

If $f(x)$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, then we can write

$$f(x) = (x - \alpha)g(x) + r$$

where $g(x)$ is a polynomial of degree $n-1$ and $r \in \mathbb{R}$. (This is a special case of "division with remainder" for polynomials.)

(a) With notation as above, show that $f(\alpha) = r$. In particular, if $f(\alpha) = 0$ then $f(x) = (x - \alpha)g(x)$.

(b) Prove the following statement by induction using part (a): If f is a polynomial of degree n then there are at most n real solutions of the equation $f(x) = 0$.

(6) Prove the following statement by induction: For all $n \in \mathbb{N}$ such that $n \geqslant 5$, $n! \geqslant 3^{n-1}$.

(7) Define a sequence $a_1, a_2, a_3, \ldots$ recursively by $a_1 = 1$, $a_2 = 7$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geqslant 3$. Prove the following statement by strong induction: For all $n \in \mathbb{N}$, $a_n = 3 \cdot 2^n - 5$.

(8) For which $c \in \mathbb{Z}$ does the equation $492x + 213y = c$ have a solution $x, y \in \mathbb{Z}$?

(9) Find all solutions $x, y \in \mathbb{Z}$ of the equation $52x + 91y = 26$.

(10) Find all solutions $x \in \mathbb{Z}$ of the congruence $42x \equiv 12 \bmod 57$ and express your answer in congruence notation.

(11) Give a proof by contradiction of the following statement: There does not exist an integer x such that $x \equiv 9 \bmod 14$ and $x \equiv 5 \bmod 21$.

(12) For $n \in \mathbb{N}$, what are the possible values of (a) $\gcd(n, n+7)$ and (b) $\gcd(7n+11, 3n+5)$? Justify your answers carefully.

(13) For $a, b \in \mathbb{N}$ we define the *least common multiple* $\operatorname{lcm}(a, b)$ of a and b to be the smallest positive integer l such that $a \mid l$ and $b \mid l$.

(a) Suppose we write a and b as products of primes:

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

and

$$b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$$

where r is a non-negative integer, $p_1, p_2, \ldots, p_r$ are distinct primes and α_i and β_i are non-negative integers for each $i = 1, 2, \ldots, r$. Determine $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ in terms of these factorizations. [Note: We allow $\alpha_i = 0$ or $\beta_i = 0$ to account for primes which divide a but not b or vice versa.]

(b) Using part (a) or otherwise, prove that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$ for all $a, b \in \mathbb{N}$.

(14) Consider the sequence of primes listed in increasing order:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

Give a proof by contradiction of the following statement: If p and q are consecutive odd primes in the list, then $p + q$ has at least 3 prime factors (not necessarily different). For example:

$$3 + 5 = 8 = 2 \cdot 2 \cdot 2$$

$$5 + 7 = 12 = 2 \cdot 2 \cdot 3$$

$$7 + 11 = 18 = 2 \cdot 3 \cdot 3$$

(15) Let A, B, and C be sets, and let P, Q, and R be the statements $(x \in A)$, $(x \in B)$, and $(x \in C)$.

(a) Express the statements $(x \in A \backslash (B \cap C))$ and $(x \in (A \backslash B) \cup (A \backslash C))$ in terms of the statements P, Q, R and logical operators.

(b) Using either truth tables or a sequence of known logical equivalences, show that the statements are logically equivalent, so that $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$.

(c) Give an alternative proof that $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$ using Venn diagrams.

[Question: What is the relation between the Venn diagrams in part (c) and the truth tables in part (b)?]

(16) Determine whether each of the following equalities hold for all sets A, B, and C. (Give a proof or a counterexample using the method of Q15ab or Q15c.)

(a) $(A\backslash B)\backslash C = A\backslash(B \cup C)$.

(b) $(A \cup B)\backslash C = A \cup (B\backslash C)$.

(17) The downtown portion of a city is a rectangular grid. How many ways are there to travel from one street corner to another, a total distance of 4 blocks south and 7 blocks west, which are as short as possible?

(18) A coin is tossed 8 times. What is the probability of getting exactly 4 heads?

(19) You are dealt two cards from a standard deck. What is the probability that at least one of the cards is an ace?

[Hint: What is the inclusion-exclusion principle? Let U be the set of possible hands and A, B, C, and D the subsets of U where one of the cards in the hand is the ace of hearts, clubs, diamonds, and spades respectively.]

(20) You roll 8 dice. What is the probability that all 6 numbers appear?

[Hint: What is the inclusion exclusion principle? Let

$$U = \{(a_1, \ldots, a_8) \mid a_i \in \{1, 2, \ldots, 6\} \text{ for each } i = 1, 2, \ldots, 8\}$$

be the set of possible outcomes, and for each $j = 1, \ldots, 6$, let $A_j \subset U$ be the subset of outcomes where the number j does *not* occur.]