Sunday 2/26/17 Math 300-3 Midterm 1 review solutions

1. a.

P	Q	P => Q
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	NOT P	(NOT P) OR Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

∴ P => Q and (NOT P) OR Q have same truth tables, so are logically equivalent.

b.

P	Q	R	P AND Q	(P AND Q) => R
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

P	Q	R	Q => R	P => (Q => R)
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

∴ (P AND Q) => R and P => (Q => R) are logically equivalent.

2 a. NOT (P AND Q) ≡ (NOT P) OR (NOT Q)

b. NOT (P OR Q) ≡ (NOT P) AND (NOT Q)

"De Morgan's laws"

c. NOT (P => Q) $\underset{1a}{\equiv}$ NOT ((NOT P) OR Q) $\underset{2b}{\equiv}$ NOT (NOT P) AND (NOT Q)

≡ P AND (NOT Q)

d. NOT $((\forall x \in U)(P(x)))$ ≡ $(\exists x \in U)($NOT$(P(x)))$

e. NOT $((\exists x \in U)(P(x)))$ ≡ $(\forall x \in U)($NOT$(P(x)))$

3. a. i. $(\exists n \in \mathbb{Z})(n^2+n \text{ is odd})$

ii $(\forall n \in \mathbb{Z})(n^2+n \text{ is even})$

iii For all integers n, n^2+n is even.

iv. <u>Proof</u>: For all integers n, n is even or odd.

If n is even, $n^2+n = n\cdot(n+1)$ is even.

(using: (*) For all integers a & b, if a is even or b is even then ab is even)

If n is odd, then $n+1$ is even and $n^2+n = n\cdot(n+1)$ is even. (using (*) again). $\square$.

b. i. $(\forall x \in \mathbb{R})((x^2 > 9) \Rightarrow (x > 3))$

ii. $(\exists x \in \mathbb{R})((x^2 > 9) \text{ AND } (x \leq 3))$ (using 2c)

iii There is a real number x such that $x^2 > 9$ and $x \leq 3$.

iv. <u>Proof</u>: Let $x = -4 \in \mathbb{R}$. then $x^2 = (-4)^2 = 16 > 9$ and $x \leq 3$. $\square$

c. i. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(y \leq x)$

ii. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y > x)$

iii. For all real numbers x, there is a real number y such that $y > x$.

iv. <u>Proof</u>: Given a real number x, let $y = x+1$, then $y > x$. $\square$.

d. i. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y^2 = x)$

ii. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(y^2 \neq x)$

iii. There is a real number x such that for all real numbers y, $y^2 \neq x$.

iv. <u>Proof</u>: Let $x = -1 \in \mathbb{R}$.

Then for all real numbers y, since $y^2 \geq 0$ we have $y^2 \neq x$

$\square$.

e. i. $(\exists x \in \mathbb{R})((x \geq 0) \text{ AND } (e^x < 1))$

ii. $(\forall x \in \mathbb{R})((x < 0) \text{ OR } (e^x \geq 1))$ (using 2a)

iii. For all real numbers x, either $x < 0$ or $e^x \geq 1$.

iv. Proof For all real numbers x either $x<0$ or $x \geqslant 0$.

If $x<0$ we are done.

If $x \geqslant 0$ we must show $e^x \geqslant 1$.

The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ is increasing

(because f is differentiable and $f'(x) = e^x > 0$ for all $x \in \mathbb{R}$)

So $x \geqslant 0 \Rightarrow e^x \geqslant e^0 = 1$. $\square$.

d. i. $(\forall n \in \mathbb{Z})\left((8 \mid n^2-1) \Rightarrow ((8 \mid n-1) \text{ OR } (8 \mid n+1))\right)$

ii. $(\exists n \in \mathbb{Z})\left((8 \mid n^2-1) \text{ AND } ((8 \nmid n-1) \text{ AND } (8 \nmid n+1))\right)$

(using 2b and 2c)

iii. There is an integer n such that 8 divides n^2-1, 8 does not divide $n-1$, and 8 does not divide $n+1$.

iv. Proof: Let $n=3$, then $8 \mid n^2-1=8$, $8 \nmid n-1=2$, $8 \nmid n+1=4$. $\square$

4 a. Proof: Assume $a \mid b$ and $a \mid c$.

So $b = qa$ and $c = sa$ for some $q, s \in \mathbb{Z}$.

Then $4b - 5c = 4qa - 5sa = (4q-5s)a$

So $4b-5c = ta$ where $t = 4q-5s \in \mathbb{Z}$.

So $a \mid 4b-5c$. $\square$.

b. Counterexample: Let $a=4$, $b=c=2$.

Then $a \mid bc$ ~~and~~ $a \nmid b$ and $a \nmid c$.

c. Counterexample: Let $a=b=c=2$.

Then $a \mid b$ and $b \mid c$ and $ab \nmid c$.

d. Proof: Assume $a^2 \mid b$ and $b^3 \mid c$.

So $b = qa^2$ and $c = sb^3$ for some $q, s \in \mathbb{Z}$.

Then $c = sb^3 = s(qa^2)^3 = sq^3a^6$ (where we used $(x^m)^n = x^{mn}$ & $(xy)^n = x^n \cdot y^n$)

So $c = t \cdot a^6$ where $t = sq^3 \in \mathbb{Z}$.

So $a^6 \mid c$. $\square$

5. Proof. Note $(n^2-1) = (n-1)\cdot(n+1)$.

So $n-1$ divides n^2-1.

Also, assuming $n>2$, we have $n-1 \neq 1$ and $n-1 \neq n^2-1$

(because $n>2 \Rightarrow n^2 > 2n > n$).

So n^2-1 is not prime. $\square$

6. a. Proof: Expand RHS

$$\begin{aligned}(x+1)\cdot(x^2-x+1) &= x\cdot(x^2-x+1) + 1\cdot(x^2-x+1) \\ &= x^3 - \cancel{x^2} + \cancel{x} + \cancel{x^2} - \cancel{x} + 1 \\ &= x^3+1. \quad \square.\end{aligned}$$

b. Proof: Using part a, we have (substituting $n=x$)

$$n^3+1 = (n+1)(n^2-n+1)$$

So $n+1$ divides n^3+1.

Also, assuming $n>1$, we have $n+1 \neq 1$ and $n+1 \neq n^3+1$

(because $n>1 \Rightarrow n^3 > 1\cdot 1\cdot n = n.$).

So n^3+1 is not prime. $\square$.

7. a) Proof: We will use the quadratic formula:-

If $a,b,c \in \mathbb{R}$, $a \neq 0$, then the equation

$ax^2+bx+c=0$ $(+)$ has a real solution x if and only if $b^2-4ac \geq 0$.

In this case, the solutions of $(+)$ are given by $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

For our example $x^2+7x+5=0$. $(++)$ $b^2-4ac = 7^2 - 4.1.5 = 49-20$

$= 29 > 0$.

So there do exists real numbers x satisfying $(++)$, and

they are given by $x = \dfrac{-7 \pm \sqrt{29}}{2}$. $\square$.

b. We will use the intermediate value theorem:

If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function and a, b, c are real numbers such that $a < b$ and $f(a) < c < f(b)$ or $f(a) > c > f(b)$,

then there is a real number t such that $a < t < b$ and $f(t) = c$.

For our example the function f is defined by $f(x) = x^3 + x$ for all $x \in \mathbb{R}$. Then f is a polynomial and so is continuous (see 131).

Let $c = 3$, $a = 0$, and $b = 2$

Then $f(a) = f(0) = 0 < c = 3$

$f(b) = f(2) = 10 > c = 3$

So, by IVT, there is a real number t such that $0 < t < 2$ and $f(t) = 3$, i.e. $t^3 + t = 3$. $\square$.

c. We will apply the IVT to the function $f(x) = e^x - x^2$.

Let $c = 0$, $a = -1$, $b = 0$.

Then $f(a) = e^{-1} - (-1)^2 = 1/e - 1 < 0 = c$ (because $e = 2.71\ldots > 1$)

and $f(b) = e^0 - 0^2 = 1 - 0 > 0 = c$

So, by IVT, there is a real number t such that $-1 < t < 0$ and $f(t) = 0$, i.e. $e^t - t^2 = 0$, or $e^t = t^2$. $\square$.

8. a. $(\forall x \in \mathbb{R})((x \text{ is rational}) \Rightarrow (x^2 \text{ is rational}))$

<u>Proof</u>: Suppose x is a real number such that x is rational. Then, by the definition of rational, $x = a/b$ for some integers a and b, where $b \neq 0$.

So $x^2 = (a/b)^2 = a^2/b^2$ is also rational. $\square$.

b.

$(\forall x \in \mathbb{R})((x \text{ is irrational}) \Rightarrow (x^2 \text{ is irrational}))$.

Counterexample: Let $x = \sqrt{2}$, then x is irrational (proved in class)
and $x^2 = (\sqrt{2})^2 = 2$ is rational.

9. a. Proof: "$\Leftarrow$" If $x=0$ then $x^5+6x^3 = 0^5+6\cdot 0^3 = 0+0=0$.
"$\Rightarrow$" If $x^5+6x^3=0$
then $x^3\cdot(x^2+6)=0$
$\Rightarrow x^3=0$ OR $(x^2+6)=0$.
$\Rightarrow x=0$ OR $x^2+6=0$.
For all real numbers x, $x^2 \geq 0$, so $x^2+6 \neq 0$.
So $x=0$. $\square$.

b. Proof: "$\Rightarrow$" Suppose a is odd.
So $a=2q+1$ for some $q\in\mathbb{Z}$.
Then $a^2-1 = (2q+1)^2-1 = 4q^2+4q+1-1$
$= 4\cdot(q^2+q)$.
So a^2-1 is divisible by 4.
"$\Leftarrow$" We will prove the contrapositive
a is even $\Rightarrow 4 \nmid a^2-1$.
Suppose a is even.
So $a=2q$, some $q\in\mathbb{Z}$
Then $a^2-1 = (2q)^2-1 = 4q^2-1. = 4\cdot(q^2-1)+3$
So a^2-1 has remainder 3 on division by 4, in particular
a^2-1 is not divisible by 4. $\square$.

10. Proof: Suppose there are real numbers x, y, z such that
① $x+y+z=1$, ② $x+2y+3z=2$, and ③ $2y+4z=3$
②−①: ④ $y+2z=1$.
③−2×④: $0=1$. # $\square$.

11. a) Proof: Suppose $\sqrt{6}$ is rational.

Then $\sqrt{6} = a/b$ where a/b is a fraction in its lowest terms.

$$\sqrt{6}\,b = a$$
$$6b^2 = a^2$$

So a^2 is even (because 6 is even).

It follows that a is even (we proved in class a^2 even $\Rightarrow$ a even)

Write $a = 2c$, some $c \in \mathbb{Z}$.

Then $6b^2 = a^2 = (2c)^2 = 4c^2$.

$$3b^2 = 2c^2$$

So $3b^2$ is even. Now 3 odd $\Rightarrow$ b^2 is even

(we proved in class xy even $\Rightarrow$ (x even) OR (y even)).

So b is even, $b = 2d$, some $d \in \mathbb{Z}$.

Now $\frac{a}{b} = \frac{2c}{2d} = \frac{c}{d}$ ※ a/b is a fraction in its lowest terms □.

b) Proof: Suppose $\sqrt{2}+\sqrt{3}$ is rational.

So $\sqrt{2}+\sqrt{3} = a/b$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.

Then $(\sqrt{2}+\sqrt{3})^2 = a^2/b^2$

$$\begin{aligned} &= (\sqrt{2})^2 + 2\cdot\sqrt{2}\cdot\sqrt{3} + \sqrt{3}^2 \\ &= 2 + 2\sqrt{6} + 3 \\ &= 5 + 2\sqrt{6}. \end{aligned}$$

So $\sqrt{6} = \frac{1}{2}\left(a^2/b^2 - 5\right) = \frac{a^2 - 5b^2}{2b^2}$ ※

$\sqrt{6}$ is irrational. □.