1. a) $A = \begin{pmatrix} s+1 & -1 \\ 2 & s+4 \end{pmatrix}$

$$\begin{aligned} \det A &= (s+1)\cdot(s+4) - (-1)\cdot 2 \\ &= s^2+5s+4+2 \\ &= s^2+5s+6. = (s+2)(s+3) \end{aligned}$$

A invertible $\iff \det A \neq 0$

$\iff s \neq -2, -3.$

In this case $A^{-1} = \dfrac{1}{s^2+5s+6}\begin{pmatrix} s+4 & 1 \\ -2 & s+1 \end{pmatrix}$

(here we have used the following formulas:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det A = ad-bc$

If $\det A \neq 0$ then A is invertible & $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.)

b) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Compute A^{-1} :-

Form matrix $(A \ I)$, row reduce to obtain matrix $(I \ B)$, then $B = A^{-1}$:-

$$\begin{array}{c} \\ -R1 \\ \\ \end{array}\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ +R2 \end{array}\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ \\ \div 2 \end{array}\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{pmatrix} \overset{-R3}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ \times -1 \\ \\ \end{array}\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \overset{-R2}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

So $A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

[Check: $A \cdot A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

$= \frac{1}{2} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$ ✓.]

c) i) If $h=0$ the 2nd row is zero so the matrix is not invertible

ii) If $h=-2$ the 1st and 2nd rows ~~are equal~~ differ by a sign (i.e. $R2 = -R1$).
So the rows are linearly dependent and the matrix is not invertible

iii) We compute

$$\det \begin{pmatrix} 0 & 1 & -1 \\ -1 & 3 & h \\ 2 & 4 & -2 \end{pmatrix} = 0 \cdot \det \begin{pmatrix} 3 & h \\ 4 & -2 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} -1 & h \\ 2 & -2 \end{pmatrix} + (-1) \cdot \det \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$$

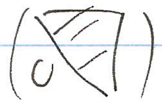
cofactor expansion along row 1

$$= -1 \cdot ((-1)\cdot(-2) - h \cdot 2) - ((-1)\cdot 4 - 3 \cdot 2)$$
$$= (2h-2) + 10$$
$$= 2h+8$$

So Matrix not invertible $\Leftrightarrow \det = 0 \Leftrightarrow h = -4$.

2. a) $\det A = \det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} = 1 \cdot 5 \cdot 8 \cdot 10 = 400$

(The determinant of an upper triangular or lower triangular matrix equals the product of the diagonal entries).



$$B = \begin{pmatrix} 1 & 3 & 9 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 2 & 4 & 1 \\ 5 & 0 & 7 & 2 \end{pmatrix}$$

$$\det B = -2 \cdot \det \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 1 \\ 5 & 0 & 2 \end{pmatrix} = -2 \cdot \left(5 \cdot \det \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \right)$$

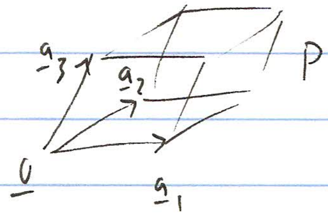
expand along row 2   expand along row 3

$$= -2 \cdot \left(5 \cdot (3\cdot 1 - 1\cdot 2) + 2 \cdot (1\cdot 2 - 3 \cdot 3) \right) = -2 \cdot (5 - 14) = 18.$$

b) A square matrix M is invertible $\iff \det M \neq 0$.
So A & B are invertible.

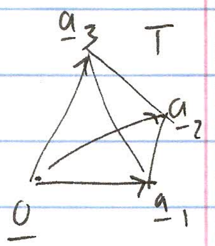
c) If A & B are invertible then so is AB,
& $(AB)^{-1} = B^{-1}A^{-1}$.
So, by part b, YES, AB is invertible.

3. a) The volume of the parallelepiped P with one vertex $\underline{0}$ and adjacent vertices $\underline{a}_1, \underline{a}_2, \underline{a}_3$ equals $|\det A|$ where $A = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3)$ is the matrix with columns $\underline{a}_1, \underline{a}_2, \underline{a}_3$
(& $|x|$ = absolute value of real number x = $\begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ (i.e. drop sign))



$$\mathrm{Vol}(P) = |\det A| = |\det(\underline{a}_1\ \underline{a}_2\ \underline{a}_3)|.$$

If T is the tetrahedron with vertices $\underline{0}, \underline{a}_1, \underline{a}_2, \underline{a}_3$ then.



$$\mathrm{Vol}(T) = \frac{1}{3} \cdot \mathrm{Area}(\mathrm{base}(T)) \cdot \mathrm{height}(T) = \frac{1}{3} \cdot \frac{1}{2} \cdot \mathrm{Area}(\mathrm{base}(P)) \cdot \mathrm{height}(P)$$
$$= \frac{1}{6} \cdot \mathrm{Vol}(P) = \frac{1}{6} \cdot |\det A|.$$

So, in our example,

$$\mathrm{Vol}(T) = \frac{1}{6} \, |\det A| = \frac{1}{6} \, |-12| = \frac{1}{6} \cdot 12 = 2.$$

b). $\mathrm{Vol}(U(T)) = |\det B| \cdot \mathrm{Vol}(T)$

$|\det B|$ is the "expansion factor" for the linear transformation $U: \mathbb{R}^3 \to \mathbb{R}^3$, $U(\underline{x}) = B \cdot \underline{x}$

$$\det B = \det \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & 5 \\ 3 & 6 & 4 \end{pmatrix}$$

$$\underset{\text{expand along row 1}}{=} -1 \cdot \det \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$$

$$= -1 \cdot (2 \cdot 4 - 5 \cdot 3) + 3 \cdot (2 \cdot 6 - 1 \cdot 3)$$

$$= 7 + 27 = 34.$$

So $\mathrm{Vol}(U(T)) = 34 \cdot 2 = 68.$

c). $C = (\underline{a}_1 - \underline{a}_2 + \underline{a}_3 \quad -\underline{a}_1 + \underline{a}_2 + \underline{a}_3 \quad \underline{a}_1 + \underline{a}_2 - \underline{a}_3)$

$$= (\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3) \cdot \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= A \cdot D$$

$$\Rightarrow \det C = \det A \cdot \det D$$

$\det A = -12$, (given)

$$\det D = \det \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \underset{\text{expand along row 1}}{=} 1 \cdot \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - (-1) \cdot \det \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= 1 \cdot (-2) + 1 \cdot 0 + 1 \cdot (-2) = -4.$$

So $\det C = (-12) \cdot (-4) = 48.$

4 a) $W = \left\{ \begin{pmatrix} x \\ x-y \\ y \end{pmatrix} \;\middle|\; x, y \text{ in } \mathbb{R} \right\}$

$= \left\{ x \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x, y \text{ in } \mathbb{R} \right\}$

$= \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right) \subset \mathbb{R}^3.$

Recall that if $\underline{v}_1, \dots, \underline{v}_p$ are vectors in $\mathbb{R}^n$, then $\mathrm{Span}(\underline{v}_1, \dots, \underline{v}_p)$ (the subset of $\mathbb{R}^n$ consisting of all linear combinations of $\underline{v}_1, \dots, \underline{v}_p$) is a subspace of $\mathbb{R}^n$.
So, in our case, W is a subspace of $\mathbb{R}^3$

$W = \mathrm{Span}(\underline{v}_1, \underline{v}_2), \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

b) $A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}$

<u>Row reduce</u>

$\rightsquigarrow \begin{array}{r} \\ +3R1 \\ +3R1 \end{array} \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{pmatrix} \rightsquigarrow \begin{array}{r} \\ \\ -2R2 \end{array} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{pmatrix}$

$\rightsquigarrow \begin{array}{r} +R3 \\ \\ \\ \end{array} \begin{pmatrix} \boxed{1} & 2 & -1 & -2 \\ 0 & \boxed{7} & 0 & -2 \\ 0 & 0 & \boxed{1} & 0 \end{pmatrix}$ REF $\rightsquigarrow \begin{array}{r} \\ \div 7 \\ \\ \end{array} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\rightsquigarrow \begin{array}{r} -2R2 \\ \\ \\ \end{array} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ RREF.

REF(A) has pivots in columns 1, 2, 3

$\Rightarrow$ Col(A) has basis $\begin{pmatrix}-3\\1\\-3\end{pmatrix}, \begin{pmatrix}1\\2\\8\end{pmatrix}, \begin{pmatrix}3\\-1\\4\end{pmatrix}$ (the corresponding columns of A)

To find basis for Nul A, use RREF(A) to solve $A\underline{x}=\underline{0}$:-

$$\begin{aligned} x_1 \quad -\tfrac{10}{7}x_4 &= 0 \\ x_2 \quad -\tfrac{2}{7}x_4 &= 0 \\ x_3 \quad &= 0 \\ x_4 \ &\text{free} \end{aligned} \Rightarrow \underline{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}\tfrac{10}{7}x_4\\\tfrac{2}{7}x_4\\0\\x_4\end{pmatrix} = x_4\begin{pmatrix}\tfrac{10}{7}\\\tfrac{2}{7}\\0\\1\end{pmatrix}$$

$\Rightarrow$ Nul A has basis $\begin{pmatrix}10/7\\2/7\\0\\1\end{pmatrix}$ $(*)$

Is $\begin{pmatrix}20\\4\\0\\14\end{pmatrix}$ in Nul A ?

Compute $A.\begin{pmatrix}20\\4\\0\\14\end{pmatrix} = \begin{pmatrix}-3 & 1 & 3 & 4\\1 & 2 & -1 & -2\\-3 & 8 & 4 & 2\end{pmatrix}\begin{pmatrix}20\\4\\0\\14\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. YES.

Alternatively, observe $\begin{pmatrix}20\\4\\0\\14\end{pmatrix} = 14.\begin{pmatrix}10/7\\2/7\\0\\1\end{pmatrix}$ so in Nul A by $(*)$ above.

5. a) Using the isomorphism
$$\begin{aligned} \mathbb{P}_2 &\longrightarrow \mathbb{R}^3 \\ \underline{p} &\longmapsto [\underline{p}]_{\mathcal{B}} \end{aligned}$$

i.e. $c_0 + c_1t + c_2t^2 \longmapsto \begin{pmatrix}c_0\\c_1\\c_2\end{pmatrix}$

it is equivalent to show that $[\underline{p}_1]_{\mathcal{B}}, [\underline{p}_2]_{\mathcal{B}}, [\underline{p}_3]_{\mathcal{B}}, [\underline{p}_4]_{\mathcal{B}}$ span $\mathbb{R}^3$.

$$[p_1]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix},\ [p_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},\ [p_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix},\ [p_4]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Form matrix with these columns & row reduce to echelon form

If have pivot in every row, then the vectors span $\mathbb{R}^3$.

$$\begin{array}{c} \div 2 \\ \\ \\ \end{array}\begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ +R1 \\ +R1 \end{array}\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 2 & 3 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \updownarrow \\ \\ \end{array}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \\ \\ -2R2 \end{array}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 3 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{3} & 0 \end{pmatrix} \quad \text{REF.}$$

So $[p_1]_{\mathcal{B}}, \ldots, [p_4]_{\mathcal{B}}$ span $\mathbb{R}^3$, & $p_1, \ldots, p_4$ span $\mathbb{P}_2$.

b). By the spanning set theorem (Thm 5, Section 4.3 of text) if V is a vector space and S is a finite set of vectors in V that spans V, then some subset of S is a basis of V. (we can remove vectors from S (one at a time) which are ~~linearly~~ linear combinations of the remaining vectors, until we are left with a subset of S that still spans and is linearly independent, so is a basis).

c) $\mathcal{B} = \{1, t+2, t^2+t+3\}$ is another basis of $\mathbb{P}_2$

$\underline{p}$ in $\mathbb{P}_2$, $[p]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad p = (-1)\cdot 1 + 2\cdot(t+2) + 1\cdot(t^2+t+3)$$
$$= 6 + 3t + t^2$$

6. a) $\mathcal{C} = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2$

We use the isomorphism
$$\mathbb{P}_2 \longrightarrow \mathbb{R}^3$$
$$p \longmapsto [p]_{\mathcal{C}}$$
i.e. $c_0 + c_1 t + c_2 t^2 \longmapsto \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$

to translate the problem to $\mathbb{R}^3$:-

$$[\underline{p}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \iff \underline{p} = c_1 \underline{b}_1 + c_2 \underline{b}_2 + c_3 \underline{b}_3$$

$$\iff [\underline{p}]_{\mathcal{E}} = c_1 [\underline{b}_1]_{\mathcal{E}} + c_2 [\underline{b}_2]_{\mathcal{E}} + c_3 [\underline{b}_3]_{\mathcal{E}}$$

i.e. $$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solve for c_1, c_2, c_3 by row reduction.

Augmented matrix:

$$\begin{array}{c} \\ -R1 \\ -R1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{array}\right) \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\sim \begin{array}{c} \\ \\ \times -1 \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -2 \end{pmatrix} \begin{array}{c} -R3 \\ \sim \\ +R3 \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \overset{\times -1}{\sim} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{array}{c} -R2 \\ \sim \end{array} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

So $$[\underline{p}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

b). $\mathbb{P}_3$ has basis $\mathcal{B} = \{1, t, t^2, t^3\}$.

Use the isomorphism
$$\mathbb{P}_3 \xrightarrow{T} \mathbb{R}^4$$
$$\underline{p} \longmapsto [\underline{p}]_{\mathcal{B}}$$
i.e. $$c_0 + c_1 t + c_2 t^2 + c_3 t^3 \longmapsto \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

to translate the problem to $\mathbb{R}^4$:

$$H = \{ \underline{p} \text{ in } \mathbb{P}_3 \mid \underline{p}(2) = 0 \}$$

$$= \{ \underline{p} = c_0 + c_1 t + c_2 t^2 + c_3 t^3 \mid c_0 + 2c_1 + 4c_2 + 8c_3 = 0 \}$$

Basis for $T(H) = \left\{ \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ in } \mathbb{R}^4 \;\middle|\; c_0 + 2c_1 + 4c_2 + 8c_3 = 0 \right\}$

$$= \mathrm{Nul}\left(\begin{pmatrix} 1 & 2 & 4 & 8 \end{pmatrix} \right) \quad :-$$

$c_0 = -2c_1 - 4c_2 - 8c_3$

c_1, c_2, c_3 free

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2c_1 - 4c_2 - 8c_3 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -8 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ $T(H)$ has basis $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ H has basis $-2 + t$, $-4 + t^2$, $-8 + t^3$.

$\dim H = \#$ vectors in basis $= 3$.