Math 235 Practice Midterm 2

Q1.

(a) Let $A = \begin{bmatrix} s+1 & -1 \\ 2 & s+4 \end{bmatrix}$ where s is a real number. Find the values of s for which A is invertible, and for these values of s compute the inverse A^{-1} of A.

(b) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Compute the inverse A^{-1} of A.

(c) In each of the following cases, find a value of h such that the matrix is *not* invertible.

(i) $\begin{bmatrix} 1 & -1 & -2 \\ 2h & h & 0 \\ 3 & 0 & -2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & h \\ 3 & 0 & 4 \end{bmatrix}$, (iii) $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 3 & h \\ 2 & 4 & -2 \end{bmatrix}$.

Q2. Consider the two 4×4 matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 & 9 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 2 & 4 & 1 \\ 5 & 0 & 7 & 2 \end{pmatrix}.$$

(a) Compute $\det A$ and $\det B$.

(b) Determine if A and B are invertible.

(c) Is AB invertible?

Q3. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ be a 3×3 matrix with $\det A = -12$.

(a) Let T be the tetrahedron in $\mathbb{R}^3$ with vertices $\mathbf{0}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. What is the volume of T?

(b) Let $B = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & 5 \\ 3 & 6 & 4 \end{bmatrix}$ and let $U \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $U(\mathbf{x}) = B\mathbf{x}$. Compute the volume of the image $U(T)$ of the tetrahedron T under the linear transformation U.

(c) Let $C = \begin{bmatrix} \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 & -\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 & \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 \end{bmatrix}$. Compute $\det C$.

Q4.

(a) Show that $W = \left\{ \begin{bmatrix} x \\ x-y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

(b) Let $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$. Find a basis of Nul($A$) and a basis of Col($A$). Is the vector $\begin{bmatrix} 20 \\ 4 \\ 0 \\ 14 \end{bmatrix}$ in Nul(A)?

Q5. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. The set $\mathcal{B} = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2$.

(a) Show that the set $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ spans $\mathbb{P}_2$ where

$$\mathbf{p}_1(t) = 2 - t - t^2, \quad \mathbf{p}_2(t) = 2t + t^2, \quad \mathbf{p}_3(t) = 3t, \quad \mathbf{p}_4(t) = 2 + t.$$

(b) Without doing any computations, explain why some subset of S is a basis for $\mathbb{P}_2$.

(c) The set $\mathcal{B} = \{1, t+2, t^2+t+3\}$ is another basis of $\mathbb{P}_2$. Find the polynomial $\mathbf{p}$ whose $\mathcal{B}$-coordinate vector is given by $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Q6.

(a) Let $\mathbb{P}_2$ be the vector space of polynomials in the variable t of degree ≤ 2. Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be the polynomials defined by

$$\mathbf{b}_1(t) = 1 + t + t^2, \ \ \mathbf{b}_2(t) = 1 + t, \ \ \mathbf{b}_3(t) = 1$$

and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Let $\mathbf{p}$ be the polynomial $\mathbf{p}(t) = 1 - t + t^2$ in $\mathbb{P}_2$. Find the $\mathcal{B}$-coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ of $\mathbf{p}$.

(b) Let $\mathbb{P}_3$ be the vector space of polynomials in the variable t of degree ≤ 3. Let H be the subspace of $\mathbb{P}_3$ that consists of the polynomials $\mathbf{p}$ such that $\mathbf{p}(2) = 0$. Find a basis for H. What is the dimension of H?