Q1 a.
$$\begin{pmatrix} 1 & 2 & 0 & 1 & 3 & 1 \\ 2 & 4 & 1 & 5 & 0 & 2 \\ 1 & 2 & -1 & -1 & 2 & 3 \end{pmatrix}$$

b.
$$\begin{array}{c} \\ -2R1 \\ -R1 \end{array}\begin{pmatrix} 1 & 2 & 0 & 1 & 3 & 1 \\ 2 & 4 & 1 & 5 & 0 & 2 \\ 1 & 2 & -1 & -1 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ +R2 \end{array}\begin{pmatrix} 1 & 2 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 & -6 & 0 \\ 0 & 0 & -1 & -2 & -1 & 2 \end{pmatrix} \rightsquigarrow \begin{array}{c} -R2 \\ -3R2 \\ \\ \end{array}\begin{pmatrix} 1 & 2 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 & -6 & 0 \\ 0 & 0 & 0 & 1 & -7 & 2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 10 & -1 \\ 0 & 0 & 1 & 0 & 15 & -6 \\ 0 & 0 & 0 & 1 & -7 & 2 \end{pmatrix}$$ reduced row echelon form.

c.
$$\begin{aligned} x_1 + 2x_2 \quad\quad + 10x_5 &= -1 \\ x_3 \quad + 15x_5 &= -6 \\ x_4 - 7x_5 &= 2. \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= -1 - 2x_2 - 10_5 \\ x_3 &= -6 - 15x_5 \\ x_4 &= 2 + 7x_5 \end{aligned}$$, x_2 & x_5 are free variables.

d.
$$\begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 3 & 1 \\ 2 & 4 & \boxed{1} & 5 & 0 & 2 \\ 1 & 2 & -1 & \boxed{-1} & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 2 & 0 & 0 & 10 & -1 \\ 0 & 0 & \boxed{1} & 0 & 15 & -6 \\ 0 & 0 & 0 & \boxed{1} & -7 & 2 \end{pmatrix}$$

augmented matrix — reduced row echelon form.

pivot positions are marked.

The pivot columns are columns 1, 3, and 4.

Q2 a. $\underline{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \underline{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \underline{v}_3 = \begin{pmatrix}1\\4\\9\end{pmatrix}$

$$A = (\underline{v}_1\ \underline{v}_2\ \underline{v}_3) = \begin{pmatrix}1&1&1\\1&2&4\\1&3&9\end{pmatrix}$$

Row reduce:

$$\begin{array}{c}\\ -R1\\ -R1\end{array}\begin{pmatrix}1&1&1\\1&2&4\\1&3&9\end{pmatrix} \rightsquigarrow \begin{array}{c}\\ \\ -2R2\end{array}\begin{pmatrix}1&1&1\\0&1&3\\0&2&8\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&1&1\\0&1&3\\0&0&2\end{pmatrix}$$

Row echelon form of A has a pivot in every row.

So the equation $A\underline{x} = \underline{b}$ has a solution for every $\underline{b}$ in $\mathbb{R}^3$

Equivalently, the equation $x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3 = \underline{b}$ has a solution for every $\underline{b}$ in $\mathbb{R}^3$.

So $\underline{v}_1, \underline{v}_2$ & v_3 span $\mathbb{R}^3$.

b. $$2x_1 + 3x_2 + 5x_3 = 0.$$

$$\Rightarrow \quad x_1 + \tfrac{3}{2}x_2 + \tfrac{5}{2}x_3 = 0,$$

$$\Rightarrow \quad x_1 = -\tfrac{3}{2}x_2 - \tfrac{5}{2}x_3,$$ x_2 & x_3 are free variables.

$$\underline{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-\frac{3}{2}x_2 - \frac{5}{2}x_3\\x_2\\x_3\end{pmatrix} = x_2\begin{pmatrix}-3/2\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-5/2\\0\\1\end{pmatrix}$$

So the plane is spanned by $\begin{pmatrix}-3/2\\1\\0\end{pmatrix}$ & $\begin{pmatrix}-5/2\\0\\1\end{pmatrix}$.

Q3. Row reduce

a). Augmented matrix:

$$\begin{pmatrix} 0 & 0 & 2 & 2 & -1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 4 \\ 2 & 2 & -2 & 0 & 4 & 6 \end{pmatrix} \rightsquigarrow \begin{array}{r} \\ \\ -2R1 \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & 2 & 2 & -1 & 1 \\ 2 & 2 & -2 & 0 & 4 & 6 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{r} \\ \\ +R2 \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & -2 & -2 & 2 & -2 \end{pmatrix} \rightsquigarrow \begin{array}{r} -R3 \\ +R3 \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{r} \\ \div 2 \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$ reduced row echelon form.

$$x_1 + x_2 + x_4 = 5$$
$$x_3 + x_4 = 0$$
$$x_5 = -1$$

$$\Rightarrow \quad x_1 = 5 - x_2 - x_4, \quad x_3 = -x_4, \quad x_5 = -1$$

, x_2 & x_4 are free variables

$$\Rightarrow \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 - x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

where x_2 & x_4 are arbitrary real numbers.

b. $$\underline{x} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$ where x_2 & x_4 are arbitrary real numbers.

Q3 c. Yes, the equation $A\underline{x} = \underline{c}$ has a solution for every $\underline{c}$ in $\mathbb{R}^3$ because the row echelon form of A has a pivot in every row

(from part (a), the reduced row echelon form of A is $\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.)

Q4. $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, where t is a real number.

a. $\underline{v}_1, \underline{v}_2$ & $\underline{v}_3$ are linearly dependent

$\Leftrightarrow$ the row echelon form of the matrix $A = (\underline{v}_1\ \underline{v}_2\ \underline{v}_3)$ does <u>not</u> have a pivot in every column.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ t & 1 & 0 \end{pmatrix} \underset{-t\cdot R1}{\leadsto} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -t \end{pmatrix} \underset{-R2}{\leadsto} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -t-1 \end{pmatrix} \quad (*)$$

So $\underline{v}_1, \underline{v}_2$ & $\underline{v}_3$ are linearly dependent $\Leftrightarrow -t-1 = 0$, i.e. $t = -1$.

In that case, solve the equation $A\underline{x} = \underline{0}$ (equivalently, $x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3 = \underline{0}$) to find a linear dependence relation:-

From $(*)$ above, if $t=-1$ then A has row echelon form $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

So $A\underline{x} = \underline{0}$ is equivalent to
$$\begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

$\therefore$ $x_1 = -x_3$, $x_2 = -x_3$, & x_3 is free

$$\underline{x} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

$(-1)\cdot\underline{v}_1 + (-1)\cdot\underline{v}_2 + 1\cdot\underline{v}_3 = \underline{0}$, i.e.,

So, a linear dependence relation is $(-1)\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (-1)\cdot\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \underline{0}$

Q4.b. $A\underline{x} = \underline{0}$ has a non trivial solution

$\iff$ the columns of A are linearly dependent.

$\iff$ $t = -1$, by part 1a).

Q5. a). Standard matrix of T $= \begin{pmatrix} T(\underline{e}_1) & T(\underline{e}_2) & T(\underline{e}_3) \end{pmatrix}$

$$= \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

b. T is onto $\iff$ the row echelon form of the matrix of T has a pivot in every row.

$$-\tfrac{1}{2}R1 \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow -R2 \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{2} & 0 & 2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So T is NOT onto (no pivot in bottom row).

c. T is one-to-one $\iff$ the row echelon form of the matrix of T has a pivot in every column.

So T is NOT one-to-one. (no pivot in last column).

d. $S: \mathbb{R}^3 \to \mathbb{R}^4$, $S(x_1, x_2, x_3) = (x_1, x_1 + x_2^2, x_1 + x_3, x_2 + x_3)$

S is NOT linear because of the x_2^2 term in the formula.

To explain carefully, recall T is linear if ① $T(c\underline{v}) = c \cdot T(\underline{v})$
② and $T(\underline{v} + \underline{w}) = T(\underline{v}) + T(\underline{w})$

for all real numbers c and vectors $\underline{v}, \underline{w}$ in $\mathbb{R}^n$ ($T: \mathbb{R}^n \to \mathbb{R}^m$).

In our case, $S(0,1,0) = (0,1,0,1)$

$S(2 \cdot (0,1,0)) = S(0,2,0) = (0,4,0,2) \neq (0,2,0,2) = 2 \cdot S(0,1,0)$

So condition ① fails for $c=2$ and $\underline{v} = (0,1,0)$.

So S is NOT linear.

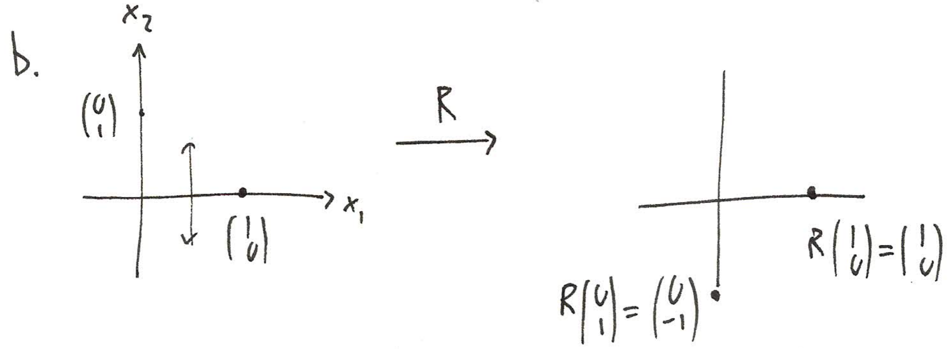
Q6. a.

$$U(\underline{x}) = T(S(\underline{x})) = B\cdot(A\cdot\underline{x}) = (B\cdot A)\cdot\underline{x}$$

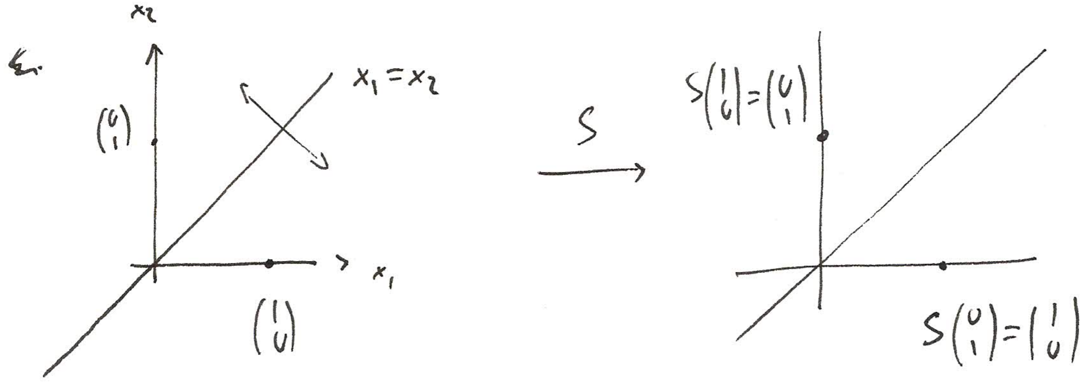
The standard matrix of U is

$$B\cdot A = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1&2\\2&0&1\end{pmatrix} = \begin{pmatrix}2&0&1\\0&1&2\end{pmatrix}.$$

b.



$\therefore$ the standard matrix of R is $\left(R\begin{pmatrix}1\\0\end{pmatrix}\ R\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1&0\\0&-1\end{pmatrix}.$



$\therefore$ the standard matrix of S is $\left(S\begin{pmatrix}1\\0\end{pmatrix}\ S\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}0&1\\1&0\end{pmatrix}.$

c. The standard matrix of V, where $V(\underline{x}) = R(S(\underline{x}))$, [Alternatively, one can argue geometrically that V is onto.]

is $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$

The row echelon form of this matrix is $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, which has a pivot in every row, so V is onto.