Math 235 Practice Midterm 1

Q1. Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & & & x_4 & + & 3x_5 & = & 1 \\ 2x_1 & + & 4x_2 & + & x_3 & + & 5x_4 & & & = & 2 \\ x_1 & + & 2x_2 & - & x_3 & - & x_4 & + & 2x_5 & = & 3 \end{array}$$

(a) Write down the augmented matrix A of the above system of linear equations.

(b) Find the reduced row echelon form of A.

(c) Find the solution set of the corresponding system of linear equations.

(d) Indicate the pivot positions in A and determine which columns of A are pivot columns.

Q2.

(a) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$?

(b) Describe the plane in $\mathbb{R}^3$ defined by the equation $2x_1 + 3x_2 + 5x_3 = 0$ as the span of a set of two vectors.

Q3. Let $A = \begin{bmatrix} 0 & 0 & 2 & 2 & -1 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & -2 & 0 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$.

(a) Find the general solution of the equation $A\mathbf{x} = \mathbf{b}$. Write your solution in vector form.

(b) (4 points) Using your answer to part (a) or otherwise, find the general solution of the equation $A\mathbf{x} = \mathbf{0}$.

(c) (4 points) Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for every vector $\mathbf{c}$ in $\mathbb{R}^3$? Justify your answer carefully.

Q4. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, where t is some real number.

(a) Find all the values of t for which $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, and find a linear dependence relation in each case.

(b) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ t & 1 & 0 \end{bmatrix}$, where t is a real number. For which values of t does the equation $A\mathbf{x} = \mathbf{0}$ have a non-trivial solution?

Q5.

(a) Find the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $T(\mathbf{e}_3) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

(b) Determine if the transformation T from part (a) is onto.

(c) Determine if the transformation T from part (a) is one-to-one

(d) Determine if the transformation $S : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$S(x_1, x_2, x_3) = (x_1, x_1 + x_2^2, x_1 + x_3, x_2 + x_3)$$

is linear. If it is linear, give its standard matrix. If it is not linear, explain which property of linear transformations it violates (and why it violates it).

Q6.

(a) Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $S(\mathbf{x}) = A\mathbf{x}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = B\mathbf{x}$, where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find the standard matrix for the linear transformation $U : \mathbb{R}^3 \to \mathbb{R}^2$, where $U(\mathbf{x}) = T(S(\mathbf{x}))$.

(b) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by reflection across the x_1-axis, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by reflection across the line $x_1 = x_2$. Find the standard matrices for R and S.

(c) For R and S defined in part (b), let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be the the linear transformation defined by $V(\mathbf{x}) = R(S(\mathbf{x}))$. Is V onto? Show your work.