235 Practice Final Exam. Solutions.

1.a.
$$\begin{array}{r} A = \\ +2R1 \\ -3R1 \\ \\ \end{array} \begin{pmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \\ 0 & 1 & -2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{r} \\ \\ \\ +R3 \end{array} \begin{pmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{row echelon form.}$$

Pivots in cols 1, 2, 3 of row echelon form of A $\Rightarrow$ basis of Col A = columns 1, 2, 3 of A

$$= \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -6 \\ -2 \end{pmatrix}$$

Basis of Row A = nonzero rows of row echelon form of A

$= (1,-2,-1,3),\ (0,1,-2,3),\ (0,0,-3,-1).$

b. A 5×9 matrix, $A \neq 0$

(i) rank A = # pivots in row echelon form of A.

\# pivots > 0 because $A \neq 0$.

\# pivots $\leq$ # rows $= 5$ & # cols $= 9$

$\therefore$ rank A = 1, 2, 3, 4 or 5.

(ii) dim Nul A = # cols − # pivots (= # free variables for $A\underline{x} = \underline{0}$)

$= 9 - \text{rank } A$

$= 8, 7, 6, 5$ or 4.

2. $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

a) Solve the characteristic equation $\det(A-\lambda I)=0$ to find the eigenvalues of A:-

$$0 = \det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = (3-\lambda)(4-\lambda) - 2\cdot 1$$

$$= 12 - 7\lambda + \lambda^2 - 2 = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5)$$

$\Rightarrow \lambda = 2, 5.$ (OR, use quadratic formula:- $a\lambda^2 + b\lambda + c = 0 \Rightarrow \lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$)

b) Given an eigenvalue λ of A, solve the equation $(A-\lambda I)\cdot \underline{x} = \underline{0}$ to find an eigenvector for this eigenvalue.

$\lambda_1 = 2.$ $\quad A - \lambda_1 I = \begin{pmatrix} 3-2 & 2 \\ 1 & 4-2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$\rightsquigarrow$ eigenvector $\underline{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, by inspection.

(or, solve using row reduction: $\begin{array}{c} \\ -R1 \end{array}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \rightsquigarrow \begin{array}{cc} x_1 & x_2 \\ \hline \end{array}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ RREF $x_1 = -2x_2$, x_2 free

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Eigenvectors are (nonzero) multiples of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$)

$\lambda_2 = 5$ $\quad A - \lambda_2 I = \begin{pmatrix} 3-5 & 2 \\ 1 & 4-5 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$

$\rightsquigarrow$ eigenvector $\underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) $P = (\underline{v}_1 \; \underline{v}_2) = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$.

$$\left(\underline{\text{CHECK}}: \quad PDP^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \right.$$

$$\left. = \frac{1}{3} \begin{pmatrix} 4 & 5 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 9 & 6 \\ 3 & 12 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = A \checkmark \right)$$

d).

$$A = PDP^{-1}$$

$$\Rightarrow A^k = PD\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}} \cdot \cancel{P}DP^{-1} \cdot\cdot \quad \cdot PDP^{-1} \qquad (k \text{ factors } A = PDP^{-1})$$

$$= PD^kP^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}^k \cdot \frac{1}{2 \cdot 1 - 1 \cdot (-1)} \begin{pmatrix} 1 & -1 \\ +1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2^k & 0 \\ 0 & 5^k \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

here, used formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$= \frac{1}{3} \begin{pmatrix} 2 \cdot 2^k & 5^k \\ -2^k & 5^k \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 \cdot 2^k + 5^k & -2 \cdot 2^k + 2 \cdot 5^k \\ -2^k + 5^k & 2^k + 2 \cdot 5^k \end{pmatrix}$$

3. $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

a) $A \cdot \underline{v} = \begin{pmatrix} 3 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 \\ 2 \cdot 1 + 4 \cdot 2 + 2 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 2 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ 6 \end{pmatrix} = 6\underline{v}$

So $\underline{v}$ is an eigenvector of A with eigenvalue $\lambda = 6$.

b). Solve $(A - \lambda I) \cdot \underline{x} = 0$ for $\lambda = 2$:-

$$A - 2I = \begin{pmatrix} 3-2 & 1 & 1 \\ 2 & 4-2 & 2 \\ 1 & 1 & 3-2 \end{pmatrix} = \begin{matrix} \\ -2R1 \\ -R1 \end{matrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ RREF}$$

$x_1 = -x_2 - x_3$, x_2 & x_3 are free. $\underline{x} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

So, the eigenspace $\mathrm{Nul}(A-\lambda I)$ for $\lambda=2$ has basis $\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}$

c) YES, A is diagonalizable because $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A (with eigenvalues $\lambda=6,2,2$).

$A = PDP^{-1}$ where $P = [\underline{v}_1 \underline{v}_2 \underline{v}_3] = \begin{pmatrix}1&-1&-1\\2&1&0\\1&0&1\end{pmatrix}$ & $D = \begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$

$= \begin{pmatrix}6&0&0\\0&2&0\\0&0&2\end{pmatrix}.$

4. $A = \begin{pmatrix}1&0&-1\\0&2&0\\4&0&5\end{pmatrix}$

a) $0 = \det(A-\lambda I) = \det\begin{pmatrix}1-\lambda&0&-1\\0&2-\lambda&0\\4&0&5-\lambda\end{pmatrix} \overset{\text{expand along row 2}}{=} +(2-\lambda)\cdot \det\begin{pmatrix}1-\lambda&-1\\4&5-\lambda\end{pmatrix}$

$= (2-\lambda)\left((1-\lambda)(5-\lambda) - (-1)\cdot 4\right) = (2-\lambda)(5-6\lambda+\lambda^2+4)$

$= (2-\lambda)(\lambda^2-6\lambda+9) = (2-\lambda)(\lambda-3)^2$

$\Rightarrow$ eigenvalues $\lambda=2$, $\lambda=3$ (w/ alg. multiplicity 2)

b). $\underline{\lambda=2}$. $A-2I = \begin{pmatrix}-1&0&-1\\0&0&0\\4&0&3\end{pmatrix} \underset{+4R1}{} \sim \begin{pmatrix}-1&0&-1\\0&0&0\\0&0&-1\end{pmatrix} \underset{\times -1}{\sim} \begin{pmatrix}-1&0&-1\\0&0&-1\\0&0&0\end{pmatrix}$

$\overset{+R2}{\sim} \begin{pmatrix}-1&0&-1\\0&0&1\\0&0&0\end{pmatrix} \overset{\times -1}{\sim} \begin{pmatrix}-1&0&0\\0&0&1\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&0&1\\0&0&0\end{pmatrix}$ RREF

$x_1 = 0,\ x_3 = 0,\ x_2$ free. So, the eigenspace

$$\underline{x} = \begin{pmatrix}0\\x_2\\0\end{pmatrix} = x_2\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Nul$(A-2I)$ for $\lambda=2$ has basis $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

$\underline{\lambda = 3.}$

$$A-3I = \begin{array}{r}\\ \\ +2R1\end{array}\begin{pmatrix}-2&0&-1\\0&-1&0\\4&0&2\end{pmatrix} \overset{\div -2}{\sim} \begin{array}{r}\\ \times -1\\ \\ \end{array}\begin{pmatrix}-2&0&-1\\0&-1&0\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&1/2\\0&1&0\\0&0&0\end{pmatrix}$$

RREF.

$$\left.\begin{array}{l} x_1 = -1/2\, x_3 \\ x_2 = 0 \\ x_3 \text{ free}\end{array}\right\} \Rightarrow \underline{x} = x_3\begin{pmatrix}-1/2\\0\\1\end{pmatrix},$$

so the eigenspace Nul$(A-3I)$ for $\lambda=3$ has basis $\begin{pmatrix}-1/2\\0\\1\end{pmatrix}$.

c). No, A is not diagonalizable, because there does not exist a basis of $\mathbb{R}^3$ consisting of eigenvectors of A :- the sum of the dimensions of the eigenspaces $= 1+1=2 < 3$.

5. $A = \begin{pmatrix}2&1\\-5&4\end{pmatrix}$

a) $0 = \det(A-\lambda I) = \det\begin{pmatrix}2-\lambda&1\\-5&4-\lambda\end{pmatrix} = (2-\lambda)(4-\lambda) - 1\cdot(-5)$

$= 8 - 6\lambda + \lambda^2 + 5 = \lambda^2 - 6\lambda + 13$

$\Rightarrow \lambda \underset{QF}{=} \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{6 \pm \sqrt{6^2 - 4.1.13}}{2.1} = \dfrac{6 \pm \sqrt{36-52}}{2}$

$= \dfrac{6 \pm \sqrt{-16}}{2} = \dfrac{6 \pm 4i}{2} = 3 \pm 2i$

b). $\lambda_1 = 3+2i \qquad A - \lambda_1 I = \begin{pmatrix}2-(3+2i)&1\\-5&4-(3+2i)\end{pmatrix} = \begin{pmatrix}-1-2i&1\\-5&1-2i\end{pmatrix}$

Eigenvector $\underline{v}_1 = \begin{pmatrix}1\\1+2i\end{pmatrix}$, by inspection (solve 1st equation, the 2nd is automatically satisfied because it's a (complex) multiple of the first eqn.)

(here we note that a solution of the equation $ax_1+bx_2=0$ is given by $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix}$.).

$\lambda_2 = 3-2i = \overline{\lambda_1}$ (complex conjugate)

$\rightsquigarrow$ eigenvector $\underline{v}_2 = \overline{\underline{v}_1} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$.

c). ~~λ~~ eigenvalue $\lambda = 3-2i = a-bi$, $a=3$, $b=2$.

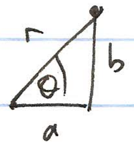
eigenvector $\underline{v} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} = \underline{u}_1 + i\underline{u}_2$, $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$\rightsquigarrow$ $A = PCP^{-1}$, $P = (\underline{u}_1 \ \underline{u}_2) = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

rotation scaling matrix.

d) $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \underbrace{r}_{\text{scale by factor } r} \cdot \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation thru angle } \theta \text{ ccw about origin}}$



$a = r\cos\theta$, $b = r\sin\theta$

$\Rightarrow$ $r = \sqrt{a^2+b^2}$ $\theta = $ "$\tan^{-1}$" (b/a)

← suitably interpreted.

Our case: $a=3, b=2$ $\Rightarrow$ $r = \sqrt{3^2+2^2} = \sqrt{13}$

$\theta = \tan^{-1}(2/3)$ $(0<\theta<\pi/2)$

6. a. Check that $\underline{v}_i \cdot \underline{v}_j = 0$ for all $i \neq j$:-

$\underline{v}_1 \cdot \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1\cdot 1 + (-1)\cdot 1 + 0\cdot(-1) = 0,$

$\underline{v}_1 \cdot \underline{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1\cdot 1 + (-1)\cdot 1 + 0\cdot 2 = 0,$ $\underline{v}_2 \cdot \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1\cdot 1 + 1\cdot 1 + (-1)\cdot 2 = 0.$

✓

So $\underline{v}_1, \underline{v}_2, \underline{v}_3$ is an orthogonal set.

An orthogonal set of nonzero vectors is linearly independent.

So $\underline{v}_1, \underline{v}_2, \underline{v}_3$ is linearly independent.

Now because $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are vectors in $\mathbb{R}^3$ and # vectors $= 3$,

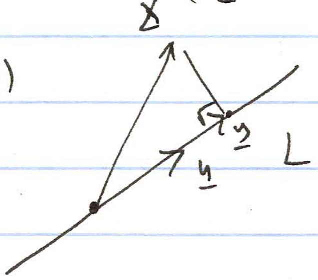
$\underline{v}_1, \underline{v}_2, \underline{v}_3$ is a basis of $\mathbb{R}^3$.

b) Because $\underline{v}_1, \underline{v}_2, \underline{v}_3$ is an orthogonal basis, we have the formula

$$\underline{x} = \left(\frac{\underline{x}\cdot\underline{v}_1}{\underline{v}_1\cdot\underline{v}_1}\right)\underline{v}_1 + \left(\frac{\underline{x}\cdot\underline{v}_2}{\underline{v}_2\cdot\underline{v}_2}\right)\underline{v}_2 + \left(\frac{\underline{x}\cdot\underline{v}_3}{\underline{v}_3\cdot\underline{v}_3}\right)\underline{v}_3$$

$$= \left(\frac{-1}{2}\right)\underline{v}_1 + \left(\frac{0}{3}\right)\underline{v}_2 + \left(\frac{9}{6}\right)\underline{v}_3 = \frac{-1}{2}\underline{v}_1 + \frac{3}{2}\underline{v}_3$$

c)



$$\underline{y} = \text{proj}_L(\underline{x}) = \left(\frac{\underline{x}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\right)\underline{u}$$

is closest point on L to $\underline{x}$

$$\underline{y} = \left(\frac{\begin{pmatrix}1\\3\\5\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}}{\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}}\right)\cdot\begin{pmatrix}1\\2\\1\end{pmatrix} = \frac{12}{6}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\4\\2\end{pmatrix}$$

$$\text{dist}(\underline{x},\underline{y}) = \|\underline{x}-\underline{y}\| = \left\|\begin{pmatrix}1\\3\\5\end{pmatrix} - \begin{pmatrix}2\\4\\2\end{pmatrix}\right\| = \left\|\begin{pmatrix}-1\\-1\\3\end{pmatrix}\right\| = \sqrt{(-1)^2+(-1)^2+3^2}$$

$$= \sqrt{11}.$$

d. $W = \text{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\5\\1\end{pmatrix}\right) \subset \mathbb{R}^3$

$$W^{\perp} = \{\underline{x} \text{ in } \mathbb{R}^3 \mid \underline{x}\cdot\underline{w} = 0 \text{ for all } w \text{ in } W\}$$

$$= \left\{\underline{x} \text{ in } \mathbb{R}^3 \;\middle|\; \underline{x}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = 0 \;\&\; \underline{x}\cdot\begin{pmatrix}2\\5\\1\end{pmatrix} = 0\right\}$$

$$= \text{Nul}\begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 1\end{pmatrix}$$

(More generally, $W = \text{Col}\,A \Rightarrow W^{\perp} = \text{Nul}(A^T)$).

$$\begin{array}{r}\\ -2R1\end{array}\begin{pmatrix}1&2&3\\2&5&1\end{pmatrix} \sim \begin{array}{r}-2R2\\ \\ \end{array}\begin{pmatrix}1&2&3\\0&1&-5\end{pmatrix} \sim \begin{pmatrix}1&0&13\\0&1&-5\end{pmatrix}$$

$x_1 = -13x_3$

$x_2 = 5x_3$

x_3 free

$\underline{x} = x_3 \begin{pmatrix}-13\\5\\1\end{pmatrix}$, $W^\perp$ has basis $\begin{pmatrix}-13\\5\\1\end{pmatrix}$.