Math 235 Practice Final

Q1.

(a) Consider the following 4×4 matrix A:

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \\ 0 & 1 & -2 & 3 \end{bmatrix}$$

Find a basis for $\mathrm{Col}(A)$ and a basis for $\mathrm{Row}(A)$.

(b) Suppose A is a 5×9 non-zero matrix. What are the possible values of (i) $\mathrm{rank}(A)$ and (ii) $\dim \mathrm{Nul}(A)$?

Q2. Consider the 2×2 matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

(a) Find all the eigenvalues of A.

(b) Find an eigenvector for each eigenvalue of A.

(c) Determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(d) Using your answer to part (c) or otherwise, determine a formula for the kth power A^k of A, valid for any positive integer k. (You should write an explicit expression for each entry of the 2×2 matrix A^k.)

Q3. Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Show that $\mathbf{v}$ is an eigenvector of A, and find the corresponding eigenvalue.

(b) Given that 2 is an eigenvalue of A, find a basis for the eigenspace of A corresponding to the eigenvalue $\lambda = 2$.

(c) Is A diagonalizable? Justify your answer carefully.

Q4. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 4 & 0 & 5 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue of A, find a basis of the eigenspace.

(c) Is A diagonalizable? Justify your answer carefully.

Q5. Let $A = \begin{bmatrix} 2 & 1 \\ -5 & 4 \end{bmatrix}$.

(a) Find the complex eigenvalues of A.

(b) Find a complex eigenvector for each complex eigenvalue of A.

(c) Find an invertible matrix P and a rotation-scaling matrix C such that $A = PCP^{-1}$.

(d) Find the scaling factor r and angle of rotation θ for the rotation scaling matrix C from part (c).

Q6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal set. Explain why $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

(b) Write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

(c) Let L be the line in $\mathbb{R}^3$ passing through the origin in the direction $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Find the closest point $\mathbf{y}$ on the line L to the point $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, and compute the distance from $\mathbf{x}$ to $\mathbf{y}$.

(d) Let $W \subset \mathbb{R}^3$ be the plane spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$. Determine a basis for the orthogonal complement $W^{\perp}$ of W.