Monday 5/4/15

Solutions to 235.5 Supplementary review Q's.

Q2. (a) $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) = 0$$

$\Rightarrow \lambda = 2,3$ eigenvalues.

$\lambda=2$. $A-2I = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow E_2 = \text{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$\lambda=3$. $A-3I = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow E_3 = \text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

(b) $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 1\cdot 2$$

$$= \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4) = 0$$

$\Rightarrow \lambda = 1, 4$ eigenvalues.

$\lambda=1$: $A-I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \Rightarrow E_1 = \text{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

$\lambda=4$: $A-4I = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \Rightarrow E_4 = \text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$

c) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 1\cdot(-1)$$

$$= \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0 \Rightarrow \lambda = 2. \text{ eigenvalue}$$

$$\lambda=2: \quad A-2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow E_2 = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix}$$

$$= +(1-\lambda)\cdot \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

Laplace expansion along row 2

$$= (1-\lambda)\cdot\left((1-\lambda)^2 - 1\right)$$

$$= (1-\lambda)\cdot(\lambda^2 - 2\lambda)$$

$$= (1-\lambda)\cdot\lambda\cdot(\lambda-2) = 0$$

$\Rightarrow$ eigenvalues $\lambda = 0, 1, 2$.

$$\lambda=0: \quad A - 0\cdot I = A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \overset{G.E.}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_0 = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right)$$

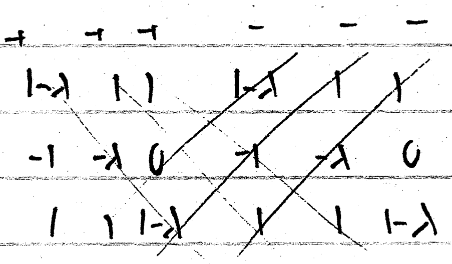
$$\lambda=1: \quad A - I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \overset{G.E.}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_1 = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right)$$

$$\lambda=2 \quad A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \overset{G.E.}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_2 = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$$

c) $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

"Sarrus' rule"



$\det(A-\lambda \underline{I}) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ -1 & -\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)\cdot(-\lambda)\cdot(1-\lambda) + 1\cdot 0\cdot 1 + 1\cdot(-1)\cdot 1 - (1-\lambda)\cdot 0\cdot 1 - 1\cdot(-1)\cdot(1-\lambda)$
$- 1\cdot(1-\lambda)\cdot 1$

~~$= \lambda^2 - \lambda^3 + 0 - 1 - 0 + 1 - \lambda + \lambda$~~

$= -\lambda\,(\lambda^2 - 2\lambda + 1) + 0 - 1 - 0 + (1-\lambda) + \lambda$

$= -\lambda\cdot(\lambda-1)^2 = 0$

$\Rightarrow \lambda = 0, 1$ eigenvalues

$\lambda = 0.$ $A - 0\cdot\underline{I} = A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \overset{G.E.}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$E_0 = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right)$

$\lambda = 1$ $A - \underline{I} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \overset{G.E.}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$E_1 = \mathrm{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right)$

d) $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Laplace expansion along 3rd row

$\det(A-\lambda\underline{I}) = \det \begin{pmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} = \left((3-\lambda)^2 - 1\right)\cdot(4-\lambda)$

$= (4-\lambda)(\lambda^2 - 6\lambda + 8)$

$= (4-\lambda)(\lambda-2)(\lambda-4)$

eigenvalues

$= -(\lambda-2)\cdot(\lambda-4)^2 = 0 \Rightarrow \lambda = 2, 4.$

$\lambda = 2$. $A - 2I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \overset{G.E.}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$,

$\Rightarrow E_2 = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right)$

$\lambda = 4$. $A - 4I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{G.E.}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_4 = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$.

Q4. a) Diagonalizable, basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$,

$\mathcal{B}$-Matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

b) Diagonalizable, basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$,

$\mathcal{B}$-matrix $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

c) NOT diagonalizable.

d) Diagonalizable, basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$,

$\mathcal{B}$-matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

e) NOT diagonalizable.

f) Diagonalizable, basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\mathcal{B}$-matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Q6.

(a) $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$A = SDS^{-1}$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \\
\mathbb{R}^2 \arrow[r, "D"] \arrow[u, "S"] & \mathbb{R}^2 \arrow[u, "S"']
\end{tikzcd}$$

$$\Rightarrow A^k = SD^kS^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2^k & 3^k \\ 0 & 3^k \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^k & -2^k+3^k \\ 0 & 3^k \end{pmatrix}$$

(b) $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ $S = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$A = SDS^{-1}$

$$\Rightarrow A^k = SD^kS^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & 4^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1}$$

Note: $(1)^k = 1$

$$= \begin{pmatrix} 1 & 4^k \\ -1 & 2 \cdot 4^k \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{1 \cdot 2 - 1 \cdot (-1)}$$

$$= \frac{1}{3} \begin{pmatrix} 2+4^k & -1+4^k \\ -2+2 \cdot 4^k & 1+2 \cdot 4^k \end{pmatrix}$$

Q7. $T: V \to V$ is <u>linear</u> if

(1) $T(f+g) = T(f) + T(g)$ for $f, g \in V$

(2) $T(cf) = cT(f)$ for $f \in V$, $c \in \mathbb{R}$.

T linear $\Rightarrow$ $T(\underline{0}) = \underline{0}$.

Q8. $T : V \to W$ linear transformation.

"rank-nullity theorem":

$$\underset{\text{"nullity" of } T}{\overset{\shortparallel}{\dim(\text{kernel}(T))}} + \underset{\text{rank of } T.}{\overset{\shortparallel}{\dim(\text{image}(T))}} = \dim(V)$$

If $T : \mathbb{R}^n \to \mathbb{R}^m$

$$\dim(\text{kernel}(T)) + \dim(\text{image}(T)) = \dim(\mathbb{R}^n) = n.$$

$\dim(\text{image}(T)) \leq m$ (because $\text{image}(T) \subset \mathbb{R}^m$)

So, if $n > m$ then

$$\dim(\text{kernel}(T)) = n - \dim(\text{image}(T)) > 0.$$

i.e. $\text{kernel}(T) \neq \{\underline{0}\}$.

Q9. V linear space, $\mathcal{B}$ a basis of V, $\mathcal{B} = (f_1, f_2, \dots, f_n)$

$T : V \to V$ linear transformation.

We have an isomorphism (invertible linear transformation)

$$L : V \longrightarrow \mathbb{R}^n \qquad L(f) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = [f]_{\mathcal{B}}$$

for $f = c_1 f_1 + c_2 f_2 + \dots + c_n f_n$

The $\mathcal{B}$-matrix M of T is defined by

$$[T(f)]_{\mathcal{B}} = M \cdot [f]_{\mathcal{B}}$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "L"'] & V \arrow[d, "L"] \\ \mathbb{R}^n \arrow[r, "M"'] & \mathbb{R}^n \end{tikzcd}$$

(a) $V = P_2 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \}$.

$T : V \to V$, $\quad T(f(x)) = f'(x) + f''(x)$.

(ii) $\mathcal{B} = (x^2, x, 1)$ is a basis of P_2

(i) T is linear: (1) $T(f+g) = (f+g)' + (f+g)'' \overset{[3]}{=} f' + g' + f'' + g''$

$= (f' + f'') + (g' + g'')$

$= T(f) + T(g)$

(2) $T(cf) = (cf)' + (cf)'' \overset{[3]}{=} cf' + cf''$

$= c \cdot (f' + f'') = c \cdot T(f)$.

(iii) $\mathcal{B}$-matrix of T.

$$\begin{tikzcd} f(x) = ax^2 + bx + c \arrow[r, "T"] \arrow[d, "L"'] & f' + f'' = 2ax + b + 2a = 2a \cdot x + (b + 2a) \arrow[d, "L"] \\ {[f(x)]_{\mathcal{B}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}} \arrow[r, "M"] & \begin{pmatrix} 0 \\ 2a \\ b+2a \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}}_{M,\ \mathcal{B}\text{-matrix}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{tikzcd}$$

(iv) T is not an isomorphism because M is not invertible.

(for example $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is in the kernel of M $\Rightarrow$ not invertible.)

(b) $V = \mathbb{R}^{2\times2}$

$T: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$, $T(X) = AX + XB$, $A = \begin{pmatrix}1&0\\2&3\end{pmatrix}$, $B = \begin{pmatrix}1&0\\1&1\end{pmatrix}$

(i) T is linear:

$$\begin{aligned}(1)\quad T(X+Y) &= A(X+Y) + (X+Y)B \qquad X, Y \in \mathbb{R}^{2\times2}\\ &= AX + AY + XB + YB\\ &= (AX+XB) + (AY+YB)\\ &= T(X) + T(Y)\end{aligned}$$

$$\begin{aligned}(2)\quad T(c\cdot X) &= A(cX) + (cX)B \qquad c \in \mathbb{R}\\ &= c(AX) + c(XB) \qquad X \in \mathbb{R}^{2\times2}\\ &= c\cdot(AX+XB) = c\cdot T(X)\end{aligned}$$

(ii) $$\mathfrak{B} = \left(\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$

is a basis of $\mathbb{R}^{2\times2}$.

(iii) $\mathfrak{B}$-matrix of T

$$X = \begin{pmatrix}a&b\\c&d\end{pmatrix} \xrightarrow{T} \begin{pmatrix}1&0\\2&3\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix} + \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}$$

$$= \begin{pmatrix}a&b\\2a+3c&2b+3d\end{pmatrix} + \begin{pmatrix}a+b&b\\c+d&d\end{pmatrix} = \begin{pmatrix}2a+b&2b\\2a+4c+d&2b+4d\end{pmatrix}$$

$$\begin{tikzcd} X \arrow[d, "L"'] \arrow[r, "T"] & \begin{pmatrix}2a+b&2b\\2a+4c+d&2b+4d\end{pmatrix} \arrow[d, "L"] \\ \begin{pmatrix}a\\b\\c\\d\end{pmatrix} \arrow[r, "M"] & \begin{pmatrix}2a+b\\2b\\2a+4c+d\\2b+4d\end{pmatrix} \end{tikzcd}$$

$$\begin{pmatrix}2a+b\\2b\\2a+4c+d\\2b+4d\end{pmatrix} = \overbrace{\begin{pmatrix}2&1&0&0\\0&2&0&0\\2&0&4&1\\0&2&0&4\end{pmatrix}}^{M\ =\ \mathfrak{B}\text{-matrix}}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

M is invertible :-

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 4 & 1 \\ 0 & 2 & 0 & 4 \end{pmatrix} \overset{G.E.}{\rightsquigarrow} \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 4 & 1 \\ 0 & 2 & 0 & 4 \end{pmatrix} \overset{G.E.}{\rightsquigarrow} \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$\Rightarrow$ rank$(M) = 4$ $\Rightarrow$ invertible.

$\therefore$ T is invertible.