235.5 Supplementary Final exam review questions

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(1) Let A be an $n \times n$ matrix. Recall that we say a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is an *eigenvector* of A with *eigenvalue* $\lambda \in \mathbb{R}$ if $A\mathbf{v} = \lambda\mathbf{v}$.

Here is the strategy to find the eigenvalues and eigenvectors of A:

(a) Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues.

(b) For each eigenvalue λ solve the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors $\mathbf{v}$ with eigenvalue λ.

[Why does this work? The equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is obtained from the equation $A\mathbf{v} = \lambda\mathbf{v}$ by rearranging the terms. This equation has a nonzero solution $\mathbf{v} \in \mathbb{R}^n$ exactly when $(A - \lambda I)$ is *not* invertible, equivalently $\det(A - \lambda I) = 0$.]

The function $\det(A - \lambda I)$ is a polynomial of degree n in the variable λ. In particular if $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

$$= (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc)$$

and we can solve the characteristic equation using the quadratic formula. If $n = 3$ we can determine the polynomial $\det(A - \lambda I)$ by computing the determinant using either Sarrus' rule or expansion along a row or column.

(2) For each of the following matrices, find all the eigenvalues and eigenvectors.

(a) $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(3) Let A be an $n \times n$ matrix. We say A is *diagonalizable* if there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of A. In this case, let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be the basis of eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the $\mathcal{B}$-matrix of the transformation $T(\mathbf{x}) = A\mathbf{x}$ is the diagonal matrix D with diagonal entries the eigenvalues $\lambda_1, \ldots, \lambda_n$ (why?). Equivalently, writing S for the matrix with columns the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, we have

$$A = SDS^{-1}.$$

We can determine whether A is diagonalizable as follows: for each eigenvalue λ, find a basis of the *eigenspace*

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda\mathbf{v}\} \subset \mathbb{R}^n$$

(the subspace of $\mathbb{R}^n$ consisting of all the eigenvectors with eigenvalue λ together with the zero vector). Now combine the bases of all the eigenspaces. These vectors are linearly independent. If there are n vectors, then they form a basis $\mathcal{B}$ of $\mathbb{R}^n$ and A is diagonalizable, otherwise A is not diagonalizable.

(4) For each of the matrices A of Q2, determine whether A is diagonalizable. If A is diagonalizable identify a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of A and write down the $\mathcal{B}$-matrix of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

(5) If A is diagonalizable we can compute an explicit formula for powers of A as follows: Write $A = SDS^{-1}$ as above where D is the diagonal matrix with diagonal entries the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any positive integer k we have

$$A^k = SD^kS^{-1}$$

(why?) and D^k is the diagonal matrix with diagonal entries $\lambda_1^k, \ldots, \lambda_n^k$.

(6) For the matrices A of Q2(a) and (b) compute a formula for A^k.

(7) Let V be a linear space and $T \colon V \to V$ a function (or transformation) from V to V. What does it mean to say that T is linear? (There are two conditions that must be satisfied.) If T is linear what is $T(\mathbf{0})$?

(8) What is the rank-nullity theorem? If $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, what can you say about the kernel of T if $n > m$?

(9) Let V be a linear space and $\mathcal{B}$ a basis of V. Let $T \colon V \to V$ be a linear transformation. What is the $\mathcal{B}$-matrix of T? In each of the following examples, (i) check that the transformation T is linear, (ii) write down a basis $\mathcal{B}$ of V, (iii) compute the $\mathcal{B}$-matrix of T, and (iv) determine whether T is an isomorphism.

  (a) $V = \mathcal{P}_2$, the linear space of polynomials $f(x)$ of degree ≤ 2, and $T \colon V \to V$, $T(f(x)) = f'(x) + f''(x)$.

  (b) $V = \mathbb{R}^{2\times 2}$, the linear space of 2×2 matrices, and $T \colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $T(X) = AX + XB$ where $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.