Monday 3/31/15.

MATH 235.5 MIDTERM 2 Review. SOLUTIONS.

A is 4×5 so $T: \mathbb{R}^5 \longrightarrow \mathbb{R}^4$, $T(\underline{x}) = A\cdot\underline{x}$

Q1. (i) $\ker(A) = \{\underline{x} \in \mathbb{R}^5 \mid A\cdot\underline{x} = \underline{0}\}$. Solve $A\cdot\underline{x} = \underline{0}$:-

$$RREF(A) = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{(given)}$$

$\rightsquigarrow$ $x_1 + 2x_2 + 3x_4 + 4x_5 = 0$

$x_3 + 3x_4 + x_5 = 0$

x_2, x_4, x_5 free variables

i.e. $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_4 - 4x_5 \\ x_2 \\ -3x_4 - x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore \left(\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right)$ is a basis of $\ker(A)$

(ii) $\text{image}(A) = \{\underline{y} \in \mathbb{R}^4 \mid \underline{y} = A\cdot\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^5\}$

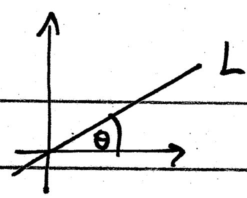
A basis for the image of A is give by the columns of A corresponding to the columns of $RREF(A)$ containing pivots (= "leading 1's" in text book).

$$RREF(A) = \begin{pmatrix} \boxed{1} & 2 & 0 & 3 & 4 \\ 0 & 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ pivots in columns 1 & 3.

$$A = \begin{pmatrix} 2 & 4 & 0 & 6 & 8 \\ 4 & 8 & 3 & 21 & 19 \\ 10 & 20 & 9 & 57 & 49 \\ 4 & 8 & 6 & 30 & 22 \end{pmatrix}$$

$\therefore$ A basis for the image of A is $\left(\begin{pmatrix} 2 \\ 4 \\ 10 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 9 \\ 6 \end{pmatrix} \right)$ (columns 1 & 3 of A).

Q2.



If L is the line through the origin in $\mathbb{R}^2$ making angle θ with the x-axis as shown, the linear transformation

$$\mathrm{Ref}_L : \mathbb{R}^2 \to \mathbb{R}^2$$

give by reflection in L has matrix $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$

In our case $\theta = 3\pi/4$

so Ref_L has matrix $\begin{pmatrix} \cos(3\pi/2) & \sin(3\pi/2) \\ \sin(3\pi/2) & -\cos(3\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Similarly for the line M $\theta = \pi/6$

so Ref_M has matrix $\begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$.

Now the matrix of the composition $T = \mathrm{Ref}_M \circ \mathrm{Ref}_L$ is given by multiplying the corresponding matrices (in the same order they are written in the composition) :-

T has matrix $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$

Q3. (i) The augmented matrix for the system of linear equations is

$$\begin{array}{r} \\ -4R1 \\ -R1 \end{array} \left(\begin{array}{ccc:c} 1 & -3 & 2 & 3 \\ 4 & -9 & 17 & 6 \\ 1 & -1 & 8 & c \end{array}\right)$$

Now use Gaussian elimination to solve:-

$$\leadsto \begin{array}{r} \\ \div 3 \\ \\ \end{array} \left(\begin{array}{ccc:c} 1 & -3 & 2 & 3 \\ 0 & 3 & 9 & -6 \\ 0 & 2 & 6 & c-3 \end{array}\right) \leadsto \begin{array}{r} \\ \\ -2R2 \end{array} \left(\begin{array}{ccc:c} 1 & -3 & 2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & c-3 \end{array}\right) \leadsto \left(\begin{array}{ccc:c} 1 & -3 & 2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & c+1 \end{array}\right)$$

End of stage 1 of algorithm. No solutions unless $c+1 = 0$, i.e., $c = -1$.

(ii) Now set $c=-1$ and solve using stage 2 of the Gaussian elimination algorithm

$$\begin{array}{c} +3R2 \\ \\ \\ \end{array}\left(\begin{array}{ccc:c} 1 & -3 & 2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc:c} 1 & 0 & 11 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

i.e. $x + 11z = -3$,
$y + 3z = -2$,
z is free variable

$$\rightsquigarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3-11z \\ -2-3z \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -11 \\ -3 \\ 1 \end{pmatrix},$$

$z \in \mathbb{R}$ is arbitary.

Q4. $\underline{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\underline{x} \in \mathbb{R}^3$ is a vector and $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^3$.

(i) $[\underline{x}]_{\mathcal{B}} = \underline{x}$ because $\mathcal{B}$ is the standard basis of $\mathbb{R}^3$

$$- \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

So $\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$$\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right]_{\mathcal{C}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Either solve by inspection OR use Gaussian elimination:—

$$\text{aug. mx} \quad \begin{array}{c} \\ -R1 \\ -R1 \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{array}\right) \rightsquigarrow \begin{array}{c} \\ \\ -R2 \end{array}\left(\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) : \begin{array}{l} c_1 = 1 \\ c_2 = -1 \\ c_3 = -1 \end{array}$$

So $\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right]_{\mathcal{C}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

(ii) $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T([\underline{y}]_{\mathcal{B}}) = [\underline{y}]_{\mathcal{C}}$$

i.e. $T(\underline{y}) = [\underline{y}]_{\mathcal{C}}$

T has <u>inverse</u> the matrix $S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

with columns the vectors of the basis $\mathcal{C}$.

So T has matrix S^{-1}.

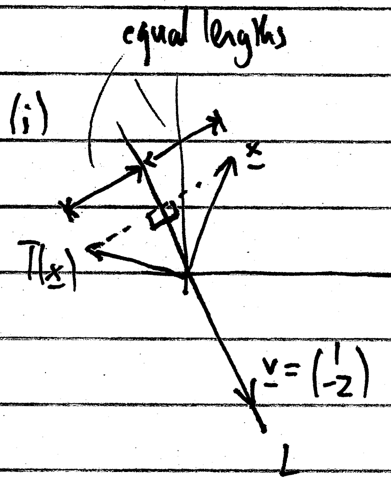
We can compute S^{-1} using Gaussian elimination.

$$\begin{array}{c} \\ -R1 \\ -R1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \begin{array}{c} \\ \\ -R2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$\therefore$ $S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ is the matrix of T.

(iii) $[\underline{x}]_{\mathcal{C}} = T([\underline{x}]_{\mathcal{B}}) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \cdot [\underline{x}]_{\mathcal{B}}$ by the definition of T.

Q5 (i)



$T: \mathbb{R}^2 \to \mathbb{R}^2$ reflection in L

$$T(\underline{x}) = 2\left(\frac{\underline{x}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\right)\underline{v} - \underline{x}$$

$$= 2\left(\frac{\binom{x}{y}\cdot\binom{1}{-2}}{\binom{1}{-2}\cdot\binom{1}{-2}}\right)\binom{1}{-2} - \binom{x}{y}$$

$$= \frac{2(x-2y)}{5}\cdot\binom{1}{-2} - \binom{x}{y} = \rightarrow$$

$$= \frac{1}{5}\begin{pmatrix} 2x-4y-5x \\ -4x+8y-5y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3x-4y \\ -4x+3y \end{pmatrix} = \underbrace{\frac{1}{5}\begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}}_{\substack{\parallel \\ A,\ \text{matrix of } T.}} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

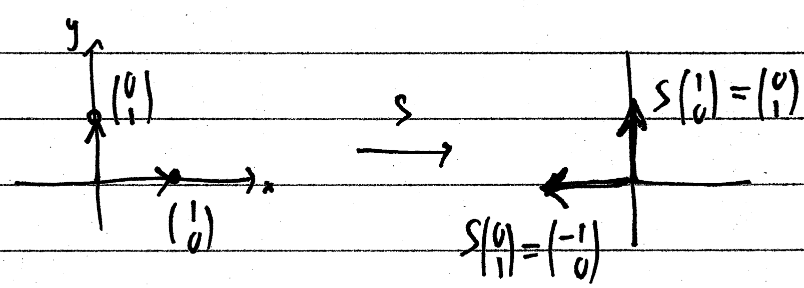
(ii) $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

counterclockwise rotation thru $\pi/2$ radians

$$\text{Matrix } B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underset{\theta = \pi/2}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Or, compute column by column

$$B = \left(S\begin{pmatrix}1\\0\end{pmatrix} \; S\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



(iii) $R = S \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

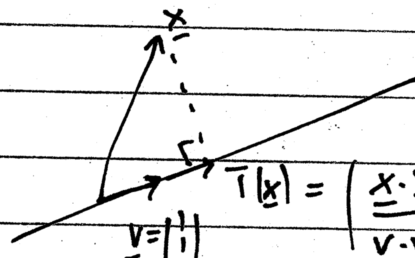
$$R(\underline{x}) = S \circ T(\underline{x}) = S(T(\underline{x})) = B \cdot (A \cdot \underline{x})$$

$$= \underbrace{BA}_{\text{matrix of } R.}\, \underline{x}$$

$$BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} +4 & -3 \\ -3 & -4 \end{pmatrix}$$

Q6.

$$\underline{x} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$



$\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad T(\underline{x}) = \left(\dfrac{\underline{x} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \right) \underline{v}$, the orthogonal projection of $\underline{x}$ onto the line L

$$T(\underline{x}) = \left(\frac{\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{7}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{x} = T(\underline{x}) + (\underline{x} - T(\underline{x}))$$

$$= \frac{7}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3} \left(\begin{pmatrix} 9 \\ 12 \\ 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix} \right) = \frac{7}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix}$$

Q7.

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & -2 \\ -12 & 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(i) $$\begin{pmatrix} 3 & -1 & -2 \\ -12 & 4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 - 1 \cdot 1 - 2 \cdot 1 \\ -12 \cdot 1 + 4 \cdot 1 + 8 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Similarly $$\begin{pmatrix} 3 & -1 & -2 \\ -12 & 4 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ are in the kernel of T.

(ii) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ are linearly independent:–

if $c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} c_1 \\ c_1 + 2c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we find $c_1 = 0$ & $c_2 = 0$.

(Or, just observe that $\begin{pmatrix}1\\1\\1\end{pmatrix}$ & $\begin{pmatrix}0\\2\\-1\end{pmatrix}$ are not parallel)

(iii) $\begin{pmatrix}-1\\4\end{pmatrix}$ is in the image of T :-

$$\begin{pmatrix}3 & -1 & -2\\ -12 & 4 & 8\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix}$$

($\begin{pmatrix}-1\\4\end{pmatrix}$ is the 2nd column of the Matrix A of T)

(iv) Rank Nullity theorem:

$T:\mathbb{R}^n \to \mathbb{R}^m$ linear transformation

$$\underbrace{\text{rank}(T)}_{\substack{\shortparallel\\ \dim(\text{image}(T))}} + \underbrace{\text{nullity}(T)}_{\substack{\shortparallel\\ \dim(\text{kernel}(T))}} = n$$

Now $\dim(\text{image}(T)) + \dim(\text{kernel}(T)) = 3$ for $T:\mathbb{R}^3 \to \mathbb{R}^2$,

$\dim(\text{image}(T)) \geq 1$ by (iii), and $\dim(\text{kernel}(T)) \geq 2$ by (ii)

$\Rightarrow \dim(\text{image}(T)) = 1$ and $\dim(\text{kernel}(T)) = 2$.

(v) In general, if $W \subset \mathbb{R}^n$ is a subspace of dimension m, and $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m \in W$ are linearly independent, then

$$\text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m) = W$$

and so $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m$ is a basis of W.

Applying this to ker(T) and image(T)
we see $\underline{v}_1, \underline{v}_2$ is a basis of ker(T)
and $\underline{w}$ is a basis of image(T).

Q8. $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}$

Compute inverse by Gaussian elimination.

$$\begin{array}{r} \\ -2R1 \\ -R1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\div(-2)} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \rightsquigarrow \begin{array}{r} \\ \\ -R2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{r} +R3 \\ +3R3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right) \rightsquigarrow \begin{array}{r} -2R2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{pmatrix}$$

Q9. Omitted (this is not on the syllabus for the exam.