1. Consider the matrix A and its reduced row-echelon form RREF(A) given below.

$$A = \begin{pmatrix} 2 & 4 & 0 & 6 & 8 \\ 4 & 8 & 3 & 21 & 19 \\ 10 & 20 & 9 & 57 & 49 \\ 4 & 8 & 6 & 30 & 22 \end{pmatrix} \qquad \mathrm{RREF}(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) Find a basis for the kernel of A.

Solution

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(ii) Find a basis for the image of A.

Solution

$$\left\{ \begin{pmatrix} 2 \\ 4 \\ 10 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 9 \\ 6 \end{pmatrix} \right\}$$

Note: To avoid having someone get this correct by accident, neither the first or third columns of RREF(A) are in im(A)

2. Let L be the line in $\mathbb{R}^2$ through the origin making angle $3\pi/4$ with the x-axis, and let M be the line in $\mathbb{R}^2$ through the origin making angle θ making angle $\pi/6$ with the x axis. Find the standard matrix for the composition $T = \mathrm{Ref}_M \circ \mathrm{Ref}_L$ of reflections through the lines L and M.

Solution:

$$L = \mathrm{span}\left\{ \begin{array}{c} \cos(3\pi/4) \\ \sin(3\pi/4) \end{array} \right\} = \mathrm{span}\left\{ \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \right\}$$

$$M = \mathrm{span}\left\{ \begin{array}{c} \cos(\pi/6) \\ \sin(\pi/6) \end{array} \right\} = \mathrm{span}\left\{ \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \right\}$$

If L is spanned by a unit vector $\mathbf{u} = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2$, then we can compute the reflection through L as

$$\mathrm{Ref}_L(\mathbf{x}) = 2(\mathbf{u}\cdot\mathbf{x})\mathbf{u} - \mathbf{x} = (2\mathrm{proj}_{\mathbf{u}} - I)\mathbf{x}\,,$$

and the matrix $(2\mathrm{proj}_{\mathbf{u}} - I)$ is given as

$$(2\mathrm{proj}_{\mathbf{u}} - I) = \begin{bmatrix} 2\cos^2\theta - 1 & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 2\sin^2\theta - 1 \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

Let A be the matrix such that $\mathrm{Ref}_L(\mathbf{x}) = A\mathbf{x}$ and let B be the matrix such that $\mathrm{Ref}_M(\mathbf{x}) = B\mathbf{x}$. We then have

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}.$$

The composition of the maps $T = \mathrm{Ref}_M \circ \mathrm{Ref}_L$ has matrix equal to the matrix product

$$BA = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

3. Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} x & - & 3y & + & 2z & = & 3 \\ 4x & - & 9y & + & 17z & = & 6 \\ x & - & y & + & 8z & = & c \end{array}$$

(i) For which values of c do the equations have a solution?
(ii) For each value of c from part (a), find all solutions of the equations.

Solution

(i) $c = -1$
(ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -11 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$, where $z \in \mathbb{R}$ is arbitrary.

4. Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ be a vector from $\mathbb{R}^3$ and

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(i) Find $[\vec{x}]_{\mathcal{B}}$ and $[\vec{x}]_{\mathcal{C}}$.
(ii) The transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T([\vec{y}]_{\mathcal{B}}) = [\vec{y}]_{\mathcal{C}},$$

for every $y \in \mathbb{R}^3$. Find the matrix of T.
(iii) Write the formula that relates $[\vec{x}]_{\mathcal{C}}$, $[\vec{x}]_{\mathcal{B}}$ and T.

Solution

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, [\vec{x}]_{\mathcal{C}} = T[\vec{x}]_{\mathcal{B}}.$$

5.

(i) Let L be the line in $\mathbb{R}^2$ spanned by the vector $v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by reflection across the line L. Recall that T is given by the formula $T(x) = 2\left(\frac{x \cdot v}{v \cdot v}\right) v - x$. Find the matrix A such that $T(x) = Ax$ (in standard coordinates).

(ii) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by counter-clockwise rotation through $\frac{\pi}{2}$ radians. Find the matrix B such that $S(x) = Bx$ (in standard coordinates).

(iii) Let $R = S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by reflection across L followed by counter-clockwise rotation through $\frac{\pi}{2}$ radians. Find the matrix C such that $R(x) = Cx$ (in standard coordinates).

6. Write the vector $\vec{x} = \langle 3, 4, 0 \rangle$ as a sum of two vectors, one of which is parallel to the line L spanned by the vector $\vec{v} = \langle 1, 1, 1 \rangle$, and the other of which is perpendicular to L.

7. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the matrix

$$A = \begin{bmatrix} 3 & -1 & -2 \\ -12 & 4 & 8 \end{bmatrix}$$

(i) Show that the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

are in the kernel of T.

(ii) Show that $\vec{v}_1, \vec{v}_2$ are linearly independent vectors. This implies $\dim(\ker T) \geq 2$.

(iii) Show that the vector

$$\vec{w} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

is in the image of T. This implies $\dim(\operatorname{im} T) \geq 1$.

(iv) Use the Rank-Nullity Theorem on the transformation T to show that

$$\dim(\ker T) = 2 \quad \text{and} \quad \dim(\operatorname{im} T) = 1.$$

(v) Explain why $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\ker T$ and $\{\vec{w}\}$ a basis for $\operatorname{im} T$.

Solution

(i) We need to show that $T(\vec{v}_1) = 0$ and $T(\vec{v}_2) = 0$. We have

$$T(\vec{v}_1) = A\vec{v}_1 = \begin{bmatrix} 3 & -1 & -2 \\ -12 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 - 1 - 2 \\ -12 + 4 + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$T(\vec{v}_2) = A\vec{v}_2 = \begin{bmatrix} 3 & -1 & -2 \\ -12 & 4 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 2 - 2 \\ 0 - 8 + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(ii) Let c_1, c_2 be scalars such that $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$. We want to show that $c_1 = c_2 = 0$. We have the system of equations:

$$c_1\vec{v}_1 + c_2\vec{v}_2 = 0 \Leftrightarrow c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 \\ c_1 - 2c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first equation is $c_1 = 0$. Then the third one becomes $c_2 = 0$. Therefore $c_1 = c_2 = 0$.

(iii) The vector $T(\vec{e}_2)$ is the first column of A, that is

$$T(\vec{e}_2) = A\vec{e}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \vec{w}.$$

Since $T(\vec{e}_2) = \vec{w}$, the vector $\vec{w}$ is in the image of T.

(iv) From the Rank-Nullity Theorem on the transformation T we know that

$$\dim(\ker T) + \dim(\operatorname{im} T) = \dim(\mathbb{R}^3) = 3.$$

From parts (b) and (c) we know that $\dim(\ker T) \geq 2$ and $\dim(\mathrm{im} T) \geq 1$. If $\dim(\ker T) > 2$ or $\dim(\operatorname{im} T) > 1$, then the sum

$$\dim(\ker T) + \dim(\operatorname{im} T)$$

is greater than 3. Hence $\dim(\ker T) = 2$ and $\dim(\operatorname{im} T) = 1$.

(v) The set $\{\vec{v}_1, \vec{v}_2\}$ consists of 2 linearly independent vectors in $\ker T$, which is of dimension 2, and so it is a basis. Similarly, the set $\{\vec{w}\}$ consists of 1 linearly independent vector in $\operatorname{im} T$, which is of dimension 1, and so it is a basis.

8. Find inverse of the matrix A if possible

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

9. Let $\mathcal{B} = \{1, x, x^2\}$, $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\}$ to be the bases of the space of polynomials of degree smaller than 2.

(i) Let us define the linear operators

$$T([\vec{y}]_{\mathcal{B}}) = [\vec{y}]_{\mathcal{C}}, S([\vec{y}]_{\mathcal{C}}) = [\vec{x}]_{\mathcal{B}}.$$

Find the matrices for S and T.

(ii) Then find the coordinates $[p]_{\mathcal{C}}$ for $p = 1 + 2x - x^2$

Solution

(i)

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$T = ([1]_{\mathcal{C}\to\mathcal{B}})^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

(ii)

$$[p]_{\mathcal{C}} = T[p]_{\mathcal{B}} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$