Q1. (8 points) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) (4 points) Give precise definitions of the kernel of T and the image of T.

(b) (4 points) Now suppose that T is invertible. What is the kernel of T? What is the image of T?

a) $\ker(T) = \{ \underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0} \}$

In words, the kernel of T is the set of all vectors $\underline{x} \in \mathbb{R}^n$ such that $T(\underline{x}) = \underline{0}$.

$\text{image}(T) = \{ \underline{y} \in \mathbb{R}^m \mid \underline{y} = T(\underline{x}) \text{ for some } \underline{x} \in \mathbb{R}^n \}$

In words, the image of T is the set of all vectors $\underline{y} \in \mathbb{R}^m$ such that $\underline{y} = T(\underline{x})$ for some $\underline{x} \in \mathbb{R}^n$. (The image of T is also called the range of T.)

b) T is invertible $\iff$ for every $\underline{y} \in \mathbb{R}^m$ there is exactly one $\underline{x} \in \mathbb{R}^n$ such that $T(\underline{x}) = \underline{y}$.

In particular T invertible $\Rightarrow$ $\text{image}(T) = \mathbb{R}^m$ and $\ker(T) = \{\underline{0}\}$

(because $\underline{0} \in \mathbb{R}^n$ is the unique vector $\underline{x}$ such that $T(\underline{x}) = \underline{0} \in \mathbb{R}^m$.)

Q2. (6 points) Let $W \subset \mathbb{R}^3$ be the plane defined by the equation

$$x + 3y + 7z = 0.$$

Find a basis of W.

$$x + 3y + 7z = 0.$$

Solve by Gaussian Elimination: –

y & z are free variables, and $x = -3y - 7z$.

i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3y - 7z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 0 \\ 1 \end{pmatrix}$ is a basis of W.

Alternatively, if $W \subset \mathbb{R}^3$ is a plane through the origin, then W is a subspace of dimension 2, and any two vectors $\underline{v}_1, \underline{v}_2 \in W$ which are not parallel give a basis of W.

c) The dimension of a subspace $W \subset \mathbb{R}^n$ is the number of vectors in a basis of the subspace. So, using (a) & (b), $\dim(\text{image}(T)) = 2$ and $\dim(\ker(T)) = 3$.

Q3. (10 points) Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ with matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & 7 & 2 \\ 1 & 3 & 1 & 2 & 3 \\ 2 & 6 & 3 & 9 & 5 \end{pmatrix}.$$

The reduced row echelon form of the matrix A is

$$\mathrm{RREF}(A) = \begin{pmatrix} 1 & 3 & 0 & -3 & 4 \\ 0 & 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(You do NOT need to check this!)

(a) (4 points) Find a basis for the image of T.

(b) (4 points) Find a basis for the kernel of T.

(c) (2 points) What is the dimension of the image of T? What is the dimension of the kernel of T?

(a) A basis for image(T) is given by the columns of A corresponding to columns of RREF(A) containing pivots (= "leading 1's" in the textbook).
In our example, this gives a basis $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}$ for image(T).

(b) Solve the equations $A\underline{x} = \underline{0}$ using RREF(A) :-

$$A\underline{x} = 0 \iff \mathrm{RREF}(A)\cdot \underline{x} = \underline{0} \iff \begin{aligned} x_1 + 3x_2 - 3x_4 + 4x_5 &= 0 \\ x_3 + 5x_4 - x_5 &= 0. \end{aligned}$$

$$\iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_2 + 3x_4 - 4x_5 \\ x_2 \\ -5x_4 + x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

x_2, x_4 & x_5 are free.

$x_2, x_4, x_5 \in \mathbb{R}$ are arbitrary.

$\therefore$ A basis for ker(T) is

$$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$