Math 235 Practice Midterm 1.

Instructions: Exam time is 2 hours. You are allowed one sheet of notes (letter-size paper, both sides). Calculators, the textbook, and additional notes are *not* allowed. Justify all your answers carefully.

Q1.

(a) Write down the augmented matrix for the following system of linear equations and find the reduced row echelon form of the matrix. Indicate which row operation you are performing at each step.

$$\begin{array}{rcrcrcrcr} x_1 & + & 5x_2 & - & x_3 & + & 3x_4 & = & 2 \\ -2x_1 & - & 10x_2 & + & 3x_3 & - & 8x_4 & = & -1 \\ 3x_1 & + & 15x_2 & - & 3x_3 & + & 9x_4 & = & 6 \end{array}$$

(b) Using the reduced row echelon matrix in your answer from Part (a):

(i) Identify the pivot positions in the augmented matrix.

(ii) Identify the basic and free variables in the linear system.

(iii) Does the system have no solutions, exactly one solution, or infinitely many solutions? Why?

Q2. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 1 \end{bmatrix}$.

(a) Find the general solution of the equation $A\mathbf{x} = \mathbf{b}$. Write your solution in vector form.

(b) Using your answer to part (a) or otherwise, find the general solution of the equation $A\mathbf{x} = \mathbf{0}$.

(c) Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for every vector $\mathbf{c}$ in $\mathbb{R}^4$? Justify your answer carefully.

Q3. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

(a) Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ linearly independent?

(b) Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbb{R}^3$?

Justify your answers carefully.

Q4.

(a) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $S(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and $S(\mathbf{e}_2) = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$. Find the standard matrix of S.

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}$, and $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $U(\mathbf{x}) = B\mathbf{x}$, where $B = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$. Find the standard matrix of the linear transformation $V\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $V(\mathbf{x}) = U(T(\mathbf{x}))$.

(c) Find the standard matrix of the linear transformation $W\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $x_2 = -x_1$.

Q5.

(a) Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 7 \\ -1 & 1 & 5 \end{bmatrix}$$

(b) Using your answer to part (a) or otherwise, solve the system of linear equations

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 7x_3 & = & 3 \\ -x_1 & + & x_2 & + & 5x_3 & = & 5 \end{array}$$