235 Final exam review questions

December 11, 2016

(1) Find the dimension of the subspace spanned by the vectors

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\4\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}.$$

(2) Find the dimension of the subspace that is the solution set of the equation $A\mathbf{x} = \mathbf{0}$ with

$$A = \begin{bmatrix}0 & 2 & 4 & 2\\1 & 3 & 7 & 4\\1 & 5 & 11 & 6\end{bmatrix}$$

(3) If a 7×4 matrix A has rank 3, find dim Nul A, dim Row A, and rank A^T.

(4) Let

$$A = \begin{bmatrix}1 & 1 & 1 & 1\\5 & 4 & 3 & 2\\1 & 2 & 3 & 4\end{bmatrix}$$

(a) i. Find a basis for Col A.
   ii. What is the dimension of Col A?
   iii. For what value of k is Col A a subspace of $\mathbb{R}^k$?

(b) i. Find a basis for Row A.
   ii. What is the dimension of Row A?
   iii. For what value of k is Row A a subspace of $\mathbb{R}^k$?

(c) i. Find a basis for Nul A.

ii. What is the dimension of Nul A?

iii. For what value of k is Nul A a subspace of $\mathbb{R}^k$?

(5) Let

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}.$$

(a) Find the eigenvalues of A and the corresponding eigenvectors.

(b) Is A diagonalizable? If so, write down an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

(6) Let

$$A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$$

(a) Find the eigenvalues of A and the corresponding eigenvectors.

(b) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{v}$ as a linear combination of the eigenvectors found in part (a).

(c) Compute the limit $\lim_{n\to\infty} A^n\mathbf{v}$.

(7) Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ -3 & 1 & 5 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Is A diagonalizable? Justify your answer carefully.

(8) Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(c) Is A diagonalizable? If so, write down an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

(9) Let

$$A = \begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix}.$$

(a) Find all the complex eigenvalues of A.

(b) For each complex eigenvalue of A, find a complex eigenvector corresponding to the eigenvalue.

(c) Find an invertible matrix P and a rotation-scaling matrix C such that $A = PCP^{-1}$.

(d) Compute the scaling factor r and the angle of rotation θ for the matrix C found in part (c).

(10) Let

$$A = \begin{bmatrix} 4 & -2 \\ 5 & -2 \end{bmatrix}.$$

(a) Find the complex eigenvalues of A and the corresponding complex eigenvectors.

(b) Find an invertible matrix P and a rotation-scaling matrix C such that $A = PCP^{-1}$.

(c) Compute the scaling factor r and the angle of rotation θ for the matrix C found in part (b).

(d) Using your answer to part (c) or otherwise, compute A^{100}.

(11) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ be vectors in $\mathbb{R}^3$ and let $\mathbf{u}_3$ be a non-zero vector in $\mathbb{R}^3$ such that $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$.

(a) Explain why the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of $\mathbb{R}^3$. (You may quote a theorem from class or the book.)

(b) Let the vector $\mathbf{y} = \begin{bmatrix} -5 \\ 5 \\ 5 \end{bmatrix}$ be written in this basis as $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$. Find c_1 and c_2.

(c) Compute the distance from $\mathbf{y}$ to the line spanned by $\mathbf{u}_2$.

(12) (a) Find a unit vector $\mathbf{u}$ in the line through the origin in $\mathbb{R}^2$ spanned by the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(b) Find an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^2$ which includes the vector $\mathbf{u}$.

(c) Find the $\mathcal{B}$-coordinate vector $[\mathbf{y}]_{\mathcal{B}}$ of the vector $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.