1. a) Augmented matrix

$$\begin{array}{c} \\ +2R1 \\ -3R1 \end{array}\begin{pmatrix} 1 & 5 & -1 & 3 & 2 \\ -2 & -10 & 3 & -8 & -1 \\ 3 & 15 & -3 & 9 & 6 \end{pmatrix} \rightsquigarrow \begin{array}{c} +R2 \\ \\ \\ \end{array}\begin{pmatrix} 1 & 5 & -1 & 3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 5 & 0 & 1 & 5 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

reduced row echelon form.

b) i) Pivot positions are $(1,1)$ & $(2,3)$
(where (i,j) means the position in row i and column j)

ii) Basic variables: x_1 & x_3
(correspond to columns of the coefficient matrix containing a pivot)
Free variables: x_2 & x_4

iii) The system is consistent (because there's no pivot in the last column of the augmented matrix), and there are free variables. So there are infinitely many solutions.

More precisely, we have

$$\begin{aligned} x_1 + 5x_2 + x_4 &= 5 \\ x_3 - 2x_4 &= 3 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x_1 &= 5 - 5x_2 - x_4 \\ x_3 &= 3 + 2x_4 \end{aligned}$$

x_2 & x_4 are free $\qquad$ x_2 & x_4 are free

In vector form, $$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5-5x_2-x_4 \\ x_2 \\ 3+2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ +2 \\ 1 \end{pmatrix}$$

where x_2 & x_4 are arbitrary real numbers.

2. a) Augmented matrix:

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 3 \\ -1 & 1 & -3 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{r} \\ -2R1 \\ -R1 \\ +R1 \end{array} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 4 \\ 1 & 1 & -1 & 3 \\ -1 & 1 & -3 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{r} \\ \\ -R2 \\ -R2 \end{array} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ row reduced echelon form.

$x_1 + x_3 = 1$ $\rightsquigarrow$ $x_1 = 1 - x_3$

$x_2 - 2x_3 = 2$ $\quad$ $x_2 = 2 + 2x_3$

x_3 is free $\quad$ x_3 is free

In vector form, $$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 - x_3 \\ 2 + 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

where x_3 is an arbitrary real number.

b) $\underline{x} = x_3 \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ where x_3 is an arbitrary real number

c) No, because the row echelon form of the coefficient matrix A does NOT have a pivot in every row.

3. a) $\underline{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix}1\\3\\6\end{pmatrix}$.

$$\begin{array}{c} \\ -R1 \\ -R1 \end{array}\begin{pmatrix}1&1&1\\1&2&3\\1&3&6\end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ -2R2 \end{array}\begin{pmatrix}1&1&1\\0&1&2\\0&2&5\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}$$

row echelon form.

The row echelon form of the matrix $A = (\underline{v}_1\ \underline{v}_2\ \underline{v}_3)$ has a pivot in every column.

So the equation $A\underline{x} = \underline{0}$ has only the trivial solution $\underline{x} = \underline{0}$ (no free variables). Equivalently, $\underline{v}_1, \underline{v}_2$ & $\underline{v}_3$ are linearly independent.

b) The row echelon form of the matrix $A = (\underline{v}_1, \underline{v}_2\ \underline{v}_3)$ has a pivot in every row. So the equation $A\underline{x} = \underline{b}$ has a solution for every $\underline{b}$ in $\mathbb{R}^3$. Equivalently, $\underline{v}_1, \underline{v}_2$ & $\underline{v}_3$ span $\mathbb{R}^3$.

4.

a)

$$S(\underline{x}) = A\cdot\underline{x}$$

where $A = \left(S(\underline{e}_1)\ S(\underline{e}_2)\right)$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 5 \end{pmatrix}$$

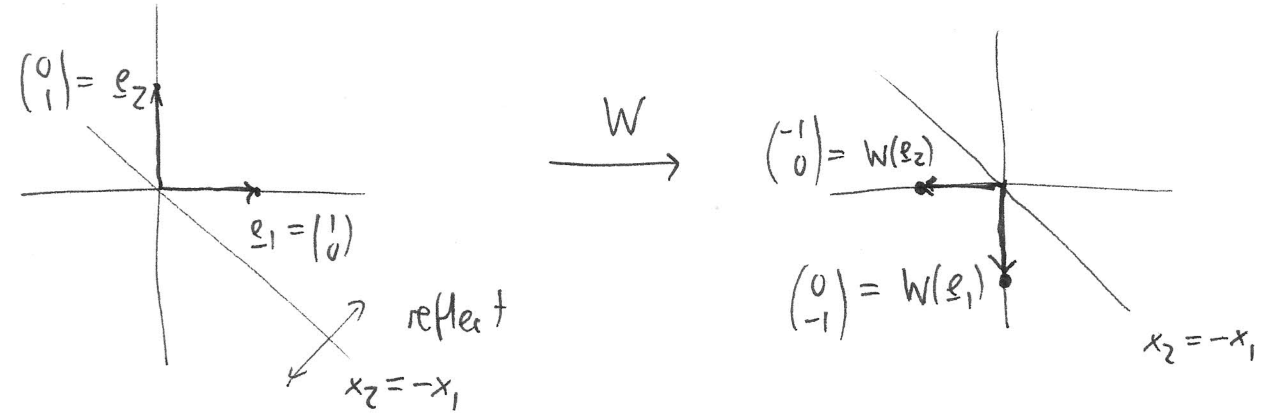
b) $V(\underline{x}) = U(T(\underline{x})) = B\cdot(A\cdot\underline{x}) = (BA)\cdot\underline{x}$

So the standard matrix of V is

$$BA = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 2\cdot1+7\cdot5 & 2\cdot4+7\cdot2 \\ 1\cdot1+3\cdot5 & 1\cdot4+3\cdot2 \end{pmatrix}$$

$$= \begin{pmatrix} 37 & 22 \\ 16 & 10 \end{pmatrix}$$

c)



So, the standard matrix of W is $\left(W(\underline{e}_1)\ W(\underline{e}_2)\right) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

5. a)

$$\begin{array}{r} \\ -2R1 \\ +R1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 7 & 0 & 1 & 0 \\ -1 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \begin{array}{r} \\ \\ -2R2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 2 & 7 & 1 & 0 & 1 \end{array}\right)$$

$$\begin{array}{r} -2R3 \\ \rightsquigarrow \\ -3R3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right) \rightsquigarrow \begin{array}{r} -R2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -9 & 4 & -2 \\ 0 & 1 & 0 & -17 & 7 & -3 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -3 & 1 \\ 0 & 1 & 0 & -17 & 7 & -3 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 8 & -3 & 1 \\ -17 & 7 & -3 \\ 5 & -2 & 1 \end{pmatrix} \qquad \left(\underline{\text{Check}}: \begin{pmatrix} 8 & -3 & 1 \\ -17 & 7 & -3 \\ 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 7 \\ -1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark \right)$$

b) The equations can be written as

$$A\underline{x} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

So the solution is $\underline{x} = A^{-1}\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 & -3 & 1 \\ -17 & 7 & -3 \\ 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ -28 \\ 9 \end{pmatrix}$