1. a) $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 5 \\ 4 & 1 & 1 \end{pmatrix}$

$$\Rightarrow \det A = 1 \cdot \det \begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix} - 3 \cdot \det \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \quad \left(\text{cofactor expansion along 1st row}\right)$$

$$= 1 \cdot (3 \cdot 1 - 5 \cdot 1) - 3 \cdot (2 \cdot 1 - 5 \cdot 4) + 2 \cdot (2 \cdot 1 - 3 \cdot 4)$$

$$= -2 - 3 \cdot (-18) + 2 \cdot (-10)$$

$$= 32.$$

b) Yes, A is invertible because $\det A \neq 0$.

2. a) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

$$\det A = 1 \cdot \det \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 0 & 1 & 0 \end{pmatrix} \quad (\text{cofactor expansion along 1st row})$$

$$= 1 \cdot (-1) \cdot \det \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \quad (\text{cofactor expansion along 3rd row})$$

$$= (-1) \cdot (2 \cdot 2 - 4 \cdot 4) = 12.$$

b) -12. (interchanging two rows multiplies the determinant by -1.)

c) -12. (same reason)

d) $10 \cdot 12 = 120$ (multiplying a row by a scalar multiplies the determinant by the same scalar)

e) 12. (adding a multiple of one row to another doesn't change the determinant)

3\. a) $$\mathrm{Vol}(S) = \left|\det\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}\right|$$

cofactor expansion along 1st row

$$= \left| 2\cdot\det\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + 1\cdot\det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right|$$

$$= \left| 2\cdot(1\cdot2 - 3\cdot1) + 1\cdot(0\cdot1 - 1\cdot1) \right|$$

$$= |-2-1| = |-3| = 3.$$

b). $$\mathrm{Vol}(T_A(S)) = |\det A|\cdot \mathrm{Vol}(S)$$

$$= \left|\det\begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 7 \\ 0 & 0 & -1 \end{pmatrix}\right| \cdot 3$$

$$= |1\cdot3\cdot-1|\cdot 3$$

$$= 3\cdot3 = 9.$$

(determinant of triangular matrix is product of diagonal entries)

4\. a) Recall: if V is a vector space, a subset $H \subset V$ of V is called a <u>subspace</u> if it satisfies the 3 conditions

1. $\underline{0}$ is in H
2. $\underline{v}, \underline{w}$ in H $\Rightarrow$ $\underline{v}+\underline{w}$ in H
3. $\underline{v}$ in H, c in $\mathbb{R}$ $\Rightarrow$ $c\underline{v}$ in H.

If $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ x,y \text{ in } \mathbb{R} \text{ and } y = x^2 \right\} \subset \mathbb{R}^2$

then H does <u>not</u> satisfy 2. For example, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are in H

but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is not in H, because $5 \neq 3^2$.

So H is not a subspace of $\mathbb{R}^2$

(Alternatively, you can show that H does not satisfy 3.)

b) $K = \{ at^2+bt+c \mid a,b,c \text{ in } \mathbb{R} \text{ and } a+b=1\} \subset \mathbb{P}_2$.

The zero vector $\underline{0}$ in $\mathbb{P}_2$ is the zero polynomial

$$\underline{0} = 0\cdot t^2 + 0\cdot t + 0.$$

We see $\underline{0}$ is not in K (because $0+0 \neq 1$)

So K is NOT a subspace of $\mathbb{P}_2$.

5. Recall: a function $T: V \longrightarrow W$ from a vector space V to a vector space W is called a <u>linear transformation</u> if it satisfies the 2 conditions

1. $T(\underline{v}+\underline{w}) = T(\underline{v}) + T(\underline{w})$ for all $\underline{v}, \underline{w}$ in V
2. $T(c\underline{v}) = cT(\underline{v})$ for $\underline{v}$ in V and c in $\mathbb{R}$

a) $T: \mathbb{P}_3 \longrightarrow \mathbb{R}^2$, $T(f(t)) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}$

1. $T(f(t)+g(t)) = \begin{pmatrix} f(1)+g(1) \\ f(2)+g(2) \end{pmatrix} = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix} + \begin{pmatrix} g(1) \\ g(2) \end{pmatrix}$

$= T(f(t)) + T(g(t))$ ✓

2. $T(c\cdot f(t)) = \begin{pmatrix} cf(1) \\ cf(2) \end{pmatrix} = c\cdot \begin{pmatrix} f(1) \\ f(2) \end{pmatrix} = c\cdot T(f(t))$ ✓

So T is a linear transformation.

b) $U: \mathbb{R}^2 \longrightarrow \mathbb{P}_2$, $U\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a^2t + b$

$$U\left(c\cdot\begin{pmatrix}a\\b\end{pmatrix}\right) = U\begin{pmatrix}ca\\cb\end{pmatrix} = (ca)^2\cdot f + cb \neq c\cdot U\begin{pmatrix}a\\b\end{pmatrix} = ca^2f + cb$$

because $(ca)^2 \neq ca^2$

So U is <u>not</u> linear.

6. a) $H = Nul(A)$ where $A = (1\ 2\ 7)$ (1×3 matrix)

$\Rightarrow H \subset R^3$ is a subspace.

b) A is in reduced row echelon form.

$A\underline{x} = \underline{0}$: $x + 2y + 7z = 0.$

$x = -2y - 7z$

y, z free variables.

i.e. $\underline{x} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2y-7z\\y\\z\end{pmatrix} = y\begin{pmatrix}-2\\1\\0\end{pmatrix} + z\begin{pmatrix}-7\\0\\1\end{pmatrix}$, y, z arbitrary real numbers.

$\therefore \begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-7\\0\\1\end{pmatrix}$ is a basis for $H = Nul(A)$

c) $\dim H = 2$

7. a) $H = \left\{ a\cdot\begin{pmatrix}1\\1\\0\\1\end{pmatrix} + b\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + c\cdot\begin{pmatrix}3\\2\\1\\2\end{pmatrix} + d\cdot\begin{pmatrix}0\\2\\0\\1\end{pmatrix} \;\middle|\; a,b,c,d \text{ in } \mathbb{R} \right\}$

$= \text{Span}\left(\begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}3\\2\\1\\2\end{pmatrix}, \begin{pmatrix}0\\2\\0\\1\end{pmatrix}\right) \subset \mathbb{R}^4$

Thus H is a subspace of $\mathbb{R}^4$.

b) $A = \begin{array}{r}\\ -R1\\ \\ -R1\end{array}\begin{pmatrix}1&1&3&0\\1&0&2&2\\0&1&1&0\\1&0&2&1\end{pmatrix} \rightsquigarrow \begin{array}{r}\\ \\ +R2\\ -R2\end{array}\begin{pmatrix}1&1&3&0\\0&-1&-1&2\\0&1&1&0\\0&-1&-1&1\end{pmatrix} \rightsquigarrow \begin{array}{r}\\ \\ \\ +\frac{1}{2}R3\end{array}\begin{pmatrix}1&1&3&0\\0&-1&-1&2\\0&0&0&2\\0&0&0&-1\end{pmatrix}$

$\rightsquigarrow \begin{pmatrix}1&1&3&0\\0&-1&-1&2\\0&0&0&2\\0&0&0&0\end{pmatrix}$ row echelon form. pivots in columns 1, 2, 4.

$\therefore$ basis of $H = \mathrm{Col}(A)$ is $\begin{pmatrix}1\\1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\0\\1\end{pmatrix}$

(columns of A corresponding to pivot columns of row echelon form).

c) $\dim H = 3$.

8. a) $$A = \begin{array}{r} \\ -R1 \\ -2R1 \end{array}\begin{pmatrix} 1 & 3 & 2 & 7 & 2 \\ 1 & 3 & 1 & 2 & 3 \\ 2 & 6 & 3 & 9 & 5 \end{pmatrix} \rightsquigarrow \begin{array}{r} \\ \\ -R2 \end{array}\begin{pmatrix} 1 & 3 & 2 & 7 & 2 \\ 0 & 0 & -1 & -5 & 1 \\ 0 & 0 & -1 & -5 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{r} \\ \times -1 \\ \\ \end{array}\begin{pmatrix} 1 & 3 & 2 & 7 & 2 \\ 0 & 0 & -1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

row echelon form

$$\overset{-2R2}{\rightsquigarrow} \begin{pmatrix} 1 & 3 & 2 & 7 & 2 \\ 0 & 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 0 & -3 & 4 \\ 0 & 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ reduced row echelon form.

$$\begin{aligned} x_1 + 3x_2 - 3x_4 + 4x_5 &= 0 \\ x_3 + 5x_4 - x_5 &= 0 \end{aligned} \rightsquigarrow \begin{aligned} x_1 &= -3x_2 + 3x_4 - 4x_5 \\ x_3 &= -5x_4 + x_5 \end{aligned}$$

x_2, x_4, x_5 free

$$\Rightarrow \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_2 + 3x_4 - 4x_5 \\ x_2 \\ -5x_4 + x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ is a basis of $\mathrm{Nul}\,A$

b) row echelon form of A has pivots in columns 1 & 3.

$\therefore \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}$ is a basis of $\mathrm{Col}(A)$

9. a) $\underline{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\underline{b}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$, $\underline{b}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$$\begin{array}{c} \\ -R1 \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 5 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ -5R2 \end{array} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$ row echelon form.

pivot in every row & column $\Rightarrow$ $\underline{b}_1, \underline{b}_2, \underline{b}_3$ is a basis of $\mathbb{R}^3$.

b) $[\underline{v}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \underline{v} = 1 \cdot \underline{b}_1 + 2 \cdot \underline{b}_2 - 1 \cdot \underline{b}_3$

$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix}$

c) $\underline{w} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, $[\underline{w}]_{\mathcal{B}} = ?$

$[\underline{w}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ where $c_1 \underline{b}_1 + c_2 \underline{b}_2 + c_3 \underline{b}_3 = \underline{w}$.

Solve for c_1, c_2, c_3:
Augmented matrix
$$\begin{array}{c} \\ -R1 \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{array}\right) \rightsquigarrow \begin{array}{c} \\ \\ -5R2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 5 & 1 & 3 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \\ \\ \div -4 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & 8 \end{array}\right) \begin{array}{c} -2R3 \\ \rightsquigarrow \\ -R3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \begin{array}{c} -R2 \\ \rightsquigarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$ $c_1 = 4, c_2 = 1, c_3 = -2$. $[\underline{w}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

10. $\underline{b}_1 = 1 + 2t + t^2$, $\underline{b}_2 = 1 + 3t + 4t^2$, $\underline{b}_3 = 1 + 4t + 8t^2$ in $\mathbb{P}_2$.

a) $\mathbb{P}_2 \xrightarrow{\sim} \mathbb{R}^3$

$c_0 + c_1 t + c_2 t^2 \longmapsto \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$, $\underline{b}_1, \underline{b}_2, \underline{b}_3 \longmapsto \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$.

$$\begin{array}{c} \\ -2R1 \\ -R1 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 8 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ -3R2 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

pivot in every row & column.

$\therefore\ \underline{b}_1, \underline{b}_2, \underline{b}_3$ is a basis of $\mathbb{P}_2$.

b) $p(t) = t^2$. $\quad [p(t)]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ where $p(t) = c_1\underline{b}_1 + c_2\underline{b}_2 + c_3\underline{b}_3$

Equivalently $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$

Solve for c_1, c_2, c_3 :-

$$\begin{array}{c} \\ -2R1 \\ -R1 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 4 & 8 & 1 \end{array}\right) \rightsquigarrow \begin{array}{c} \\ \\ -3R2 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 7 & 1 \end{array}\right) \begin{array}{c} -R3 \\ \rightsquigarrow \\ -2R3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\overset{-R2}{\rightsquigarrow}\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$[p(t)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$