Final review Solutions

1. $$A = \begin{array}{c} \\ -2R1 \\ -3R1 \\ -4R1 \\ -5R1 \end{array}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 4 & 0 & 4 & 1 \\ 5 & 1 & 4 & 1 \end{pmatrix}$$

dim Col(A) = # pivots in row echelon form of A = 3.

$$\rightsquigarrow \begin{array}{c} \\ \\ -2R2 \\ -4R2 \\ -4R2 \end{array}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -4 & 4 & 1 \\ 0 & -4 & 4 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ \\ \\ -R3 \end{array}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

row echelon form.

2. $$A = \begin{pmatrix} 0 & 2 & 4 & 2 \\ 1 & 3 & 7 & 4 \\ 1 & 5 & 11 & 6 \end{pmatrix}$$

dim Nul A = # columns − # pivots = 4 − 2 = 2.

$$\rightsquigarrow \begin{array}{c} \\ \\ -R1 \end{array}\begin{pmatrix} 1 & 3 & 7 & 4 \\ 0 & 2 & 4 & 2 \\ 1 & 5 & 11 & 6 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ -R2 \end{array}\begin{pmatrix} 1 & 3 & 7 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 4 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 7 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

row echelon form.

3. dim Nul A = # columns − rank(A) = 4 − 3 = 1

dim Row A = # pivots = rank(A) = 3

rank(A^T) = dim(Col(A^T)) = dim(Row(A)) = 3.

4. (a) $$A = \begin{array}{c} \\ -5R1 \\ -R1 \end{array}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ \\ +R2 \end{array}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

row echelon form

(i) $\begin{pmatrix}1\\5\\1\end{pmatrix}, \begin{pmatrix}1\\4\\2\end{pmatrix}$ (columns of A corresponding to pivot columns of row echelon form)

(ii) dim Col(A) = 2

(iii) $Col(A) \subset \mathbb{R}^3$, $k=3$.

(b) (i) $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\-2\\-3\end{pmatrix}$ (nonzero rows of row echelon form of A)

(ii) dim Row(A) = 2

(iii) $Row(A) \subset \mathbb{R}^4$, $k=4$

(c) (i) $\begin{array}{c} \\ \times(-1) \\ \\ \end{array}\begin{pmatrix}1&1&1&1\\0&-1&-2&-3\\0&0&0&0\end{pmatrix} \overset{-R2}{\rightsquigarrow} \begin{pmatrix}1&1&1&1\\0&1&2&3\\0&0&0&0\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{pmatrix}$

$x_1 - x_3 - 2x_4 = 0$
$x_2 + 2x_3 + 3x_4 = 0$
x_3, x_4 free

$\rightsquigarrow \underline{x} = \begin{pmatrix}x_3+2x_4\\-2x_3-3x_4\\x_3\\x_4\end{pmatrix} = x_3\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + x_4\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$

x_3, x_4 arbitrary

So $\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$ is basis of Nul A

(ii) dim Nul(A) = 2

(iii) $Nul(A) \subset \mathbb{R}^4$, $k=4$.

5. $A = \begin{pmatrix} 3 & 1 \\ 4 & 6 \end{pmatrix}$

(a) $0 = \det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 \\ 4 & 6-\lambda \end{pmatrix}$

$= (3-\lambda)(6-\lambda) - 1 \cdot 4$

$= \lambda^2 - 9\lambda + 18 - 4 = \lambda^2 - 9\lambda + 14$

$= (\lambda - 2)(\lambda - 7)$

$\Rightarrow \lambda = 2, 7$

$\underline{\lambda = 2}$. $A - 2I = \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}$, eigenspace $\text{Span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$

$\underline{\lambda = 7}$ $A - 7I = \begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}$, eigenspace $\text{Span}\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right)$

(b) YES. $P = \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$

((Columns of P are eigenvectors, Diagonal entries of D are corresponding eigenvalues)

6. $A = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$

(a) $0 = \det(A - \lambda I) = \lambda^2 - (1.2)\lambda + 0.2$

$= \frac{1}{5}(5\lambda^2 - 6\lambda + 1)$

$= \frac{1}{5}(5\lambda - 1)(\lambda - 1)$

$\Rightarrow \lambda = \underset{\underset{0.2}{\parallel}}{\frac{1}{5}}, 1.$

$\underline{\lambda = \frac{1}{5}}$: $A - (0.2)I = \begin{pmatrix} 0.6 & 0.6 \\ 0.2 & 0.2 \end{pmatrix}$, eigenspace $\text{Span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$

$\underline{\lambda = 1}$: $A - I = \begin{pmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$, eigenspace $\text{Span}\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$

(b) $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{-1}{4}\begin{pmatrix} -5 \\ -3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5 \\ 3 \end{pmatrix}.$

$$\underline{v} = \tfrac{5}{4}\begin{pmatrix}-1\\1\end{pmatrix} + \tfrac{3}{4}\begin{pmatrix}3\\1\end{pmatrix}$$

(c) $$A^n\underline{v} = \tfrac{5}{4}\cdot A^n\cdot\begin{pmatrix}-1\\+1\end{pmatrix} + \tfrac{3}{4}\cdot A^n\cdot\begin{pmatrix}3\\1\end{pmatrix}$$

$$= \tfrac{5}{4}(0.2)^n\begin{pmatrix}-1\\1\end{pmatrix} + \tfrac{3}{4}(1)^n\begin{pmatrix}3\\1\end{pmatrix}$$

$$\to \tfrac{3}{4}\cdot\begin{pmatrix}3\\1\end{pmatrix} \quad \text{as } n\to\infty$$

7. $$A = \begin{pmatrix}1 & 3 & 1\\0 & 5 & 0\\-3 & 1 & 5\end{pmatrix}$$

(a)

$$0=\det(A-\lambda I) = \det\begin{pmatrix}1-\lambda & 3 & 1\\0 & 5-\lambda & 0\\-3 & 1 & 5-\lambda\end{pmatrix}$$

$$= +(5-\lambda)\cdot\det\begin{pmatrix}1-\lambda & 1\\-3 & 5-\lambda\end{pmatrix} \qquad \text{(cofactor expansion along row 2.)}$$

$$= (5-\lambda)\big((1-\lambda)(5-\lambda)+3\big)$$

$$= (5-\lambda)(\lambda^2-6\lambda+8)$$

$$= -(\lambda-5)(\lambda-2)(\lambda-4) \quad \Rightarrow \quad \lambda=2,4,5$$

(b) Yes, because A has 3 distinct eigenvalues.

8. $$A = \begin{pmatrix}2 & 1 & -1\\-1 & 0 & 1\\1 & 1 & 0\end{pmatrix}$$

(a) $$0=\det(A-\lambda I) = \det\begin{pmatrix}2-\lambda & 1 & -1\\-1 & -\lambda & 1\\1 & 1 & -\lambda\end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix}-\lambda & 1\\1 & -\lambda\end{pmatrix} - 1\cdot\det\begin{pmatrix}-1 & 1\\1 & -\lambda\end{pmatrix} + (-1)\cdot\det\begin{pmatrix}-1 & -\lambda\\1 & 1\end{pmatrix} \qquad \left(\begin{matrix}\text{cofactor}\\\text{expansion}\\\text{along row 1.}\end{matrix}\right)$$

$$= (2-\lambda)(\lambda^2-1) - (\lambda-1) - (-1+\lambda)$$
$$= -\lambda^3+2\lambda^2+\lambda-2 \quad -2\lambda+2$$
$$= -\lambda^3+2\lambda^2-\lambda$$
$$= -\lambda(\lambda^2-2\lambda+1)$$
$$= -\lambda(\lambda-1)^2$$
$$\Rightarrow \lambda = 0, 1.$$

(b). $\underline{\lambda=0}$. $A-0\cdot I = A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{c} \\ +R1 \\ -2R1 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$

$\rightsquigarrow \begin{array}{c} \\ \\ +R2 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \overset{-R2}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ eigenspace $= \mathrm{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right)$, basis $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\underline{\lambda=1}$ $A-I = \begin{array}{c} \\ +R1 \\ -R1 \end{array}\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1+x_2-x_3=0$, $\underline{x} = \begin{pmatrix} -x_2+x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
x_2, x_3 free

eigenspace has basis $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(c) Yes, because $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a basis of eigenvectors.

$P = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

9. $A = \begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix}$

(a) $0 = \det(A - \lambda I) = \lambda^2 - 2\lambda + 10$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 10}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

(b) $\underline{\lambda = 1+3i}$ $\quad A - (1+3i)I = \begin{pmatrix} 1-3i & 5 \\ -2 & -1-3i \end{pmatrix}$

$\rightsquigarrow$ eigenspace $\operatorname{Span}\left(\begin{pmatrix} -5 \\ 1-3i \end{pmatrix}\right)$

$\underline{\lambda = 1-3i} = \overline{1+3i}$ $\rightsquigarrow$ eigenspace $\operatorname{Span}\left(\overline{\begin{pmatrix} -5 \\ 1-3i \end{pmatrix}}\right) = \operatorname{Span}\left(\begin{pmatrix} -5 \\ 1+3i \end{pmatrix}\right)$

(here bar denotes complex conjugation, $\overline{a+bi} = a-bi$)

(c) $\lambda = 1-3i = a-bi$, $a=1$, $b=3$

Eigenvector $\begin{pmatrix} -5 \\ 1+3i \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} i$

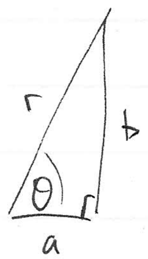
$\rightsquigarrow A = PCP^{-1}$, $P = \begin{pmatrix} -5 & 0 \\ 1 & 3 \end{pmatrix}$, $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

(d) $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\Rightarrow r = \sqrt{a^2+b^2} = \sqrt{10}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(3)$

(counterclockwise)



10. $A = \begin{pmatrix} 4 & -2 \\ 5 & -2 \end{pmatrix}$

(a) $0 = \det(A - \lambda I) = \lambda^2 - 2\lambda + 2$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\underline{\lambda = 1+i}$: $A - (1+i)I = \begin{pmatrix} 3-i & -2 \\ 5 & -3-i \end{pmatrix}$ $\leadsto$ eigenspace $\operatorname{Span}\left(\begin{pmatrix} 2 \\ 3-i \end{pmatrix}\right)$

$\underline{\lambda = 1-i = \overline{1+i}}$: eigenspace $\operatorname{Span}\left(\overline{\begin{pmatrix} 2 \\ 3-i \end{pmatrix}}\right) = \operatorname{Span}\left(\begin{pmatrix} 2 \\ 3+i \end{pmatrix}\right)$

b) $\lambda = a - bi = 1 - i \Rightarrow a = b = 1$

Eigenvector $\begin{pmatrix} 2 \\ 3+i \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$

$\leadsto A = PCP^{-1}$, $P = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

c) $r = \sqrt{a^2 + b^2} = \sqrt{2}$, $\theta = \tan^{-1}(b/a) = \tan^{-1}(1) = \pi/4$ (counterclockwise)

d) $A^{100} = (PCP^{-1})^{100} = PC^{100}P^{-1}$

C = rotation by $\pi/4$ counterclockwise & scaling by $\sqrt{2}$

$\Rightarrow C^{100}$ = rotation by $100 \cdot \pi/4$ ccw & scaling by $\sqrt{2}^{100}$

$100 \cdot \pi/4 = 25\pi = \pi + 12 \cdot (2\pi)$, $\quad \sqrt{2}^{100} = 2^{50}$

= rotation by π & scaling by 2^{50}

$= 2^{50}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -2^{50} \cdot I$

Now $A^{100} = P \cdot C^{100} \cdot P^{-1} = -2^{50} \cdot I = \begin{pmatrix} -2^{50} & 0 \\ 0 & -2^{50} \end{pmatrix}$

11.

(a) $\underline{u}_1, \underline{u}_2, \underline{u}_3$ is an orthogonal set of nonzero vectors in $\mathbb{R}^3$

$\Rightarrow$ linearly independent.

And any set of n linearly independent vectors in $\mathbb{R}^n$ also spans $\mathbb{R}^n$, so is a basis of $\mathbb{R}^n$.

(b) Because $\underline{u}_1, \underline{u}_2, \underline{u}_3$ are orthogonal,

$$c_i = \underline{y}\cdot\underline{u}_i \Big/ \underline{u}_i\cdot\underline{u}_i \quad \text{for each } i = 1,2,3$$

So
$$c_1 = \begin{pmatrix}-5\\5\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix} \Big/ \begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix} = 0.$$

$$c_2 = \begin{pmatrix}-5\\5\\5\end{pmatrix}\cdot\begin{pmatrix}0\\2\\1\end{pmatrix} \Big/ \begin{pmatrix}0\\2\\1\end{pmatrix}\cdot\begin{pmatrix}0\\2\\1\end{pmatrix} = 15/5 = 3.$$

(c) Let $L = \mathrm{Span}(\underline{u}_2)$

The closest point on L to the point $\underline{y}$ is

$$\mathrm{proj}_L(\underline{y}) = \left(\underline{y}\cdot\underline{u}_2 \Big/ \underline{u}_2\cdot\underline{u}_2\right)\cdot\underline{u}_2 = c_2\underline{u}_2 = 3\underline{u}_2$$

So, the distance from $\underline{y}$ to L is

$$\|\underline{y} - \mathrm{proj}_L(\underline{y})\| = \|\underline{y} - 3\underline{u}_2\| = \left\|\begin{pmatrix}-5\\5\\5\end{pmatrix} - 3\begin{pmatrix}0\\2\\1\end{pmatrix}\right\| = \left\|\begin{pmatrix}-5\\-1\\2\end{pmatrix}\right\|$$

$$= \sqrt{(25+1+4)} = \sqrt{30}.$$

12. (a) $\underline{u} = \underline{v} / \|\underline{v}\| = \begin{pmatrix}-1\\2\end{pmatrix}\Big/\sqrt{5}$

(b) Let $\underline{w} = \begin{pmatrix}2\\1\end{pmatrix}$, then $\underline{u}\cdot\underline{w} = 0$.

Now $\underline{u}_1 = \underline{u}$, $\underline{u}_2 = \underline{w}/\|\underline{w}\| = \begin{pmatrix}2\\1\end{pmatrix}\Big/\sqrt{5}$ is an orthonormal basis of $\mathbb{R}^2$.

(c) $\underline{y} = c_1 \underline{u}_1 + c_2 \underline{u}_2$

where $c_1 = \underline{y} \cdot \underline{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{5}{\sqrt{5}}$

$c_2 = \underline{y} \cdot \underline{u}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{5}}$

(using $\mathcal{B}$ orthonormal)

So $[\underline{y}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 5 \\ 5 \end{pmatrix}$