Sunday 12/2/18. FINAL EXAM. SPRING 2016. SOLUTIONS

MATH 132

1.

$x=6.$

$x=r\cos\theta$ $\Big\}$ $r\cos\theta = 6,\quad r=\dfrac{6}{\cos\theta} = 6\sec\theta.$ $\boxed{D}$

2. $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^n \cdot \frac{x^n}{n^2 \cdot 5^n}}_{a_n}$

Ratio test: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2\cdot 5^{n+1}} \cdot \frac{n^2\cdot 5^n}{x^n}\right|$

$\displaystyle = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}\cdot\frac{|x|}{5} = \lim_{n\to\infty}\frac{1}{(1+1/n)^2}\cdot\frac{|x|}{5} = 1\cdot\frac{|x|}{5} = \frac{|x|}{5}$

$\therefore$ absolutely convergent for $\frac{|x|}{5}<1$, i.e. $|x|<5$. divergent for $\frac{|x|}{5}>1$ i.e. $|x|>5$.

$\leadsto$ radius of convergence $R=5$. $\boxed{C}$.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{x^{n-1}}{3^n} = \sum_{n=1}^{\infty}\frac{1}{3}\cdot\left(\frac{x}{3}\right)^{n-1}$ geometric series, common ratio $r=\frac{x}{3}$.

$\Rightarrow$ convergent for $|x/3|<1$, i.e. $-3<x<3$.

divergent for $|x/3|\geq 1$. $\boxed{A}$.

4. $\displaystyle\int \ln(2x)\,dx = x\cdot\ln(2x) - \int x\cdot\frac{1}{x}\,dx = x\ln(2x) - \int dx$

$= x\ln(2x) - x + C.$ $\boxed{B}$

Integration by parts.

$\int u\,dv = uv - \int v\,du$

$u=\ln(2x)$, $dv = dx$

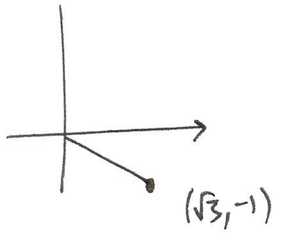
$\leadsto du = \frac{1}{2x}\cdot 2\cdot dx$ $\quad v=\int dx = x.$

$= \frac{1}{x}dx$

5. $(x,y) = (\sqrt{3},-1)$. $r=\sqrt{x^2+y^2} = 2.$

$x = r\cos\theta \quad \leadsto \quad \cos\theta = x/r = \sqrt{3}/2$

$y = r\sin\theta \qquad \sin\theta = y/r = -1/2$



$\Rightarrow \quad \theta = -\pi/6.$

$(r,\theta) = (2,-\pi/6) \qquad \boxed{D}.$

(Remark: Other representations of the same point can be obtained using

$(r,\theta) = (r,\theta + 2\pi\cdot k)$, k an integer

& $(r,\theta) = (-r, \theta+\pi)$.).

6. a.

$$\int_0^1 x^3(1+x^4)^4\,dx = \int_0^1 \tfrac{1}{4}(1+x^4)^4 \cdot 4x^3dx = \int_1^2 \tfrac{1}{4}u^4\,du$$

$u = 1+x^4$

$\cancel{u=x^4}$ $du = 4x^3dx$

$$= \left[\tfrac{1}{4}\cdot\tfrac{1}{5}u^5\right]_1^2 = 1/20\cdot(32-1)$$

$$= 31/20.$$

b.

$$f(x) = \frac{x^2}{(1+4x^3)^2} = x^2\cdot\frac{1}{(1-(-4x^3))^2} \overset{\text{subst. } y=-4x^3 \text{ in } (*)}{=} x^2\cdot\sum_{n=0}^{\infty}(n+1)\cdot(-4x^3)^n$$

$(*)$ $\frac{1}{(1-y)^2} = \sum_{n=0}^{\infty}(n+1)y^n$, valid for $|y|<1$

obtained by differentiating the geometric series

$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$, valid for $|y|<1$

$$= x^2\cdot\sum_{n=0}^{\infty}(n+1)\cdot(-1)^n\cdot 4^n\cdot x^{3n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot(n+1)\cdot 4^n\cdot x^{3n+2}$$

valid for $|-4x^3|<1$, i.e., $|x| < \frac{1}{4^{1/3}}$

7. a. $f(x) = \ln(1+x)$, $a=1$.

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -1\cdot(1+x)^{-2}$

$$f'''(x) = (-1)\cdot(-2)\cdot(1+x)^{-3}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)\cdot(-2)\cdots(-(n-1))\cdot(1+x)^{-n}$$

$$= (-1)^{n-1}(n-1)!\cdot(1+x)^{-n}$$

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!\,2^{-n}$$

(for $n \geq 1$.)

Taylor series: $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\underset{a=1}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}\cdot(x-1)^n$$

$$= f(1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \ln(1+1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!\cdot 2^{-n}}{n!}(x-1)^n$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\cdot\frac{1}{2^n}\cdot(x-1)^n.$$

Alternative solution.

$$\ln(1+x) = \ln(2+(x-1)) = \ln\left(2\cdot\left(1+\frac{x-1}{2}\right)\right) = \ln 2 + \ln\left(1+\left(\frac{x-1}{2}\right)\right)$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot\left(\frac{x-1}{2}\right)^n}{n} = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\cdot\frac{1}{2^n}\cdot(x-1)^n,$$

valid for $\left|\frac{x-1}{2}\right| < 1$

i.e. $|x-1| < 2$.

substitute $y = \frac{x-1}{2}$ in series

$$\ln(1+y) = \sum_{n=1}^{\infty} (-1)^{n-1}\cdot\frac{y^n}{n}, \text{ valid for } |y|<1$$

obtained by integrating the geometric series

$$\frac{1}{1+y} = \frac{1}{1-(-y)} = \sum_{n=0}^{\infty}(-y)^n = \sum_{n=0}^{\infty}(-1)^n\cdot y^n, \text{ valid for } |y|<1.$$

7b.

Given: $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$

$$\Rightarrow \quad f(x) = 9x \tan^{-1}(4x^3) = 9x \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(4x^3)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 9 \cdot 4^{2n+1}}{2n+1} \cdot x^{6n+3+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 9 \cdot 4^{2n+1}}{2n+1} \cdot x^{6n+4}$$

8a. $\sum_{n=0}^{\infty} (-1)^n \cdot \underbrace{\frac{(5x)^n}{3\sqrt{n}+2}}_{a_n}$

Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(5x)^{n+1}}{3\sqrt{n+1}+2} \cdot \frac{3\sqrt{n}+2}{(5x)^n} \right|$

$$= \lim_{n\to\infty} 5 \cdot |x| \cdot \frac{3\sqrt{n}+2}{3\sqrt{n+1}+2} = \lim_{n\to\infty} 5 \cdot |x| \cdot \frac{3 + 2/\sqrt{n}}{3 \cdot \sqrt{1+1/n} + 2/\sqrt{n}}$$

$$= 5 \cdot |x| \cdot \left(\frac{3+0}{3 \cdot 1 + 0}\right) = 5 \cdot |x| < 1 \iff |x| < 1/5$$

So, the series is absolutely convergent for $|x| < 1/5$, divergent for $|x| > 1/5$, radius of convergence $R = 1/5$.

Interval of convergence: check endpoints $x = \pm 1/5$.

$x = -1/5$: $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(-1)^n}{3\sqrt{n}+2} = \sum_{n=0}^{\infty} \frac{1}{3\sqrt{n}+2}$

Limit comparison test: $\lim_{n\to\infty} \frac{\frac{1}{3\sqrt{n}+2}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{3\sqrt{n}+2} = \lim_{n\to\infty} \frac{1}{3 + 2/\sqrt{n}} = \frac{1}{3} \neq 0$.

$\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges (p-series, $p = 1/2 \leq 1$) $\underset{LCT}{\Rightarrow}$ $\sum_{n=0}^{\infty} \frac{1}{3\sqrt{n}+2}$ diverges.

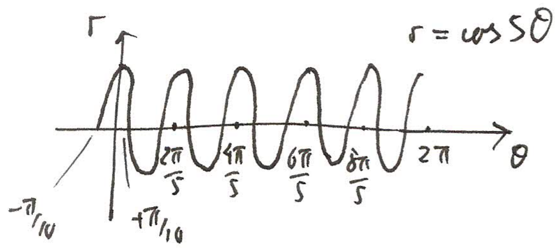
$x = 1/5$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(1)^n}{3\sqrt{n}+2} = \sum_{n=0}^{\infty} (-1)^n \cdot \underbrace{\frac{1}{3\sqrt{n}+2}}_{b_n}$

Alternating series, b_n positive, decreasing, $b_n \to 0$ as $n \to \infty$ ✓.

$\therefore$ convergent by alternating series test.

So, interval of convergence is $-1/5 < x \leq \frac{1}{5}$, i.e. $(-1/5, 1/5]$.

8b. $r = \cos(5\theta)$



$\cos 5\theta = 0 \iff 5\theta = \pm\pi/2 + (2\pi)\cdot k$

k integer

$\iff \theta = \pm\pi/10 + (2\pi/5)\cdot k$

$\therefore$ Right hand loop corresponds to $-\pi/10 \leq \theta \leq \pi/10$.

$$\text{Area} = \int_{-\pi/10}^{\pi/10} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \cdot \int_{-\pi/10}^{\pi/10} (\cos 5\theta)^2 \, d\theta = \frac{1}{2} \cdot \int_{-\pi/10}^{\pi/10} \frac{1}{2}(1 + \cos(10\theta)) \, d\theta$$

$\cos 2t = 2(\cos t)^2 - 1$

$\Rightarrow (\cos t)^2 = (1 + \cos 2t)/2$

$\Rightarrow (\cos 5\theta)^2 = (1 + \cos(10\theta))/2$

$$= \frac{1}{4} \cdot \int_{-\pi/10}^{\pi/10} 1 + \cos(10\theta) \, d\theta$$

$$= 1/4 \cdot \left[\theta + \frac{1}{10} \sin(10\theta) \right]_{-\pi/10}^{\pi/10}$$

$$= 1/4 \cdot \left((\pi/10 + 0) - (-\pi/10 + 0) \right)$$

$$= 1/4 \cdot \pi/5 = \boxed{\pi/20}.$$

9a.

$$x = \frac{1}{2}t^2 \qquad \Big| \quad 0 \leq t \leq 4.$$
$$y = \frac{1}{3}(2t+1)^{3/2}$$

Length of this parametrized curve $= \displaystyle\int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^4 \sqrt{(t)^2 + \left(\frac{1}{3}\cdot\frac{3}{2}\cdot(2t+1)^{1/2}\cdot 2\right)^2}\, dt$$

$$= \int_0^4 \sqrt{t^2 + (2t+1)}\, dt$$

$$= \int_0^4 \sqrt{(t+1)^2}\, dt$$

$$= \int_0^4 t+1 \, dt$$

$$= \left[\,t^2/2 + t\,\right]_0^4$$

$$= (16/2 + 4) - (0+0) = \boxed{12}.$$

b. $x = \sec t$

$y = \tan t$

eq of tangent at $t = \pi/6$?

Slope $m = \left.\dfrac{dy}{dx}\right|_{t=\pi/6} = \left.\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=\pi/6} = \left.\dfrac{(\sec t)^2}{\sec t \cdot \tan t}\right|_{t=\pi/6} = \left.\dfrac{1}{\sin t}\right|_{t=\pi/6}$

$$= 1/(1/2) = 2.$$

$t = \pi/6 \Rightarrow (x,y) = (\sec(\pi/6), \tan(\pi/6)) = \left(1/\tfrac{\sqrt{3}}{2},\ 1/\sqrt{3}\right) = \left(2/\sqrt{3},\ 1/\sqrt{3}\right)$

$\therefore$ eq. of tangent $(y - 1/\sqrt{3}) = 2\cdot(x - 2/\sqrt{3})$, $y = 2x + \left(\frac{1}{\sqrt{3}} - \frac{4}{\sqrt{3}}\right)$

$y = 2x - \frac{3}{\sqrt{3}}$, $\boxed{y = 2x - \sqrt{3}}$.