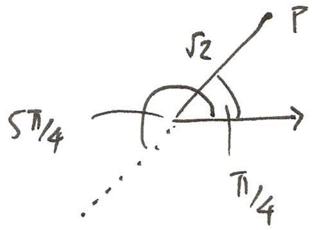
Sunday 12/2/18. FINAL EXAM. SPRING 2015. SOLUTIONS.
MATH 132

1. $(r,\theta) = (-\sqrt{2}, 5\pi/4) = (\sqrt{2}, \pi/4)$



$\Rightarrow (x,y) = (\sqrt{2}\cos\pi/4, \sqrt{2}\sin\pi/4) = (1,1).$ $\boxed{A}$.

2. For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ one of the following is true:

- the series converges for $|x-a| < R$ & diverges for $|x-a| > R$, for some $R > 0$.
- the series converges when $x = a$ & diverges when $x \neq a$.
- the series converges for all x.

So the impossible result is $\boxed{D}$

3. $\sum_{n=1}^{\infty} \underbrace{n!\,(3x-1)^n}_{a_n}$

Ratio test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,(3x-1)^{n+1}}{n!\,(3x-1)^n}\right| = \lim_{n\to\infty} (n+1)\cdot|3x-1| = \begin{cases} \infty \text{ if } |3x-1| > 0 \\ 0 \text{ if } |3x-1| = 0. \end{cases}$

So, the series converges for $3x-1 = 0$, i.e., $x = 1/3$

& diverges for $x \neq 1/3$.

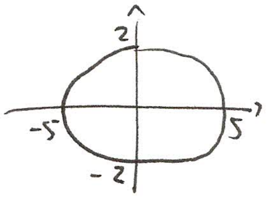
Interval of convergence $= \{1/3\}$ $\boxed{D}$.

4. $x = 5\sin t \qquad y = 2\cos t.$

$$\Rightarrow \quad \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = (\sin t)^2 + (\cos t)^2 = 1.$$

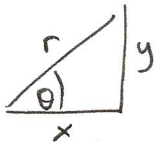
Equation $\quad \dfrac{x^2}{25} + \dfrac{y^2}{4} = 1 \quad . \qquad \boxed{B}$

(This is an ellipse.)



5. $3r\cos\theta + 4r\sin\theta = 1$

$x = r\cos\theta, \; y = r\sin\theta \quad \rightsquigarrow \quad 3x + 4y = 1. \qquad \boxed{C}$



6. a

$$\int x^2\sin(x^3)\,dx = \int \tfrac{1}{3}\sin(x^3)\cdot 3x^2\,dx = \int \tfrac{1}{3}\sin u\,du$$

$u = x^3 \quad du = 3x^2dx$

$$= -\tfrac{1}{3}\cos u + C = -1/3\cos(x^2) + C.$$

b. $\sin(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$

$$\Rightarrow \quad f(x) = 2x^3\sin(2\pi x^2) = 2x^3 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2\pi x^2)^{2n+1}}{(2n+1)!} = 2x^3 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \overbrace{(2\pi)^{2n+1}}^{2^{2n+1}\cdot\pi^{2n+1}}\, x^{4n+2}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+2} \cdot \pi^{2n+1}}{(2n+1)!} \cdot x^{4n+5}$$

7a. $$\sum_{n=0}^{\infty} \underbrace{(-1)^n \cdot \frac{(5x-3)^n}{3n+2}}_{a_n}$$

Ratio test : $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(5x-3)^{n+1}}{3(n+1)+2} \cdot \frac{3n+2}{(5x-3)^n} \right|$$

$$= \lim_{n\to\infty} \frac{3n+2}{3n+5} \cdot |5x-3| = \lim_{n\to\infty} \frac{3+2/n}{3+5/n} \cdot |5x-3| = |5x-3|.$$

So, absolutely convergent for $|5x-3| < 1$ i.e. $-1 < 5x-3 < 1$

$+2 < 5x < 4$

$2/5 < x < 4/5$

(& divergent for $x < 2/5$, $x > 4/5$)

Check endpoints: $(5x-3) = \pm 1$

$(5x-3) = -1$: $$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(-1)^n}{3n+2} = \sum_{n=0}^{\infty} \frac{1}{3n+2}$$

limit comparison test : $$\lim_{n\to\infty} \frac{1/(3n+2)}{1/n} = \lim_{n\to\infty} \frac{n}{3n+2}$$

$$= \lim_{n\to\infty} \frac{1}{3+2/n} = 1/3 \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (p-series, $p = 1 \leq 1$) $\Rightarrow$ $\sum_{n=0}^{\infty} \frac{1}{3n+2}$ divergent.

$(5x-3) = +1$: $$\sum_{n=0}^{\infty} (-1)^n \cdot \underbrace{\frac{1}{3n+2}}_{b_n}$$ b_n positive, decreasing,

$\lim_{n\to\infty} b_n = 0$.

$\Rightarrow$ convergent by alternating series test.

So, interval of convergence : $2/5 < x \leq 4/5$, i.e. $(2/5, 4/5]$.

7b.

$$f(x) = \frac{x^6}{(1-4x)^2} = x^6 \cdot \frac{1}{(1-4x)^2} \underset{\uparrow}{=} x^6 \cdot \sum_{n=0}^{\infty} (n+1)\cdot(4x)^n$$

$$\frac{1}{(1-y)^2} \underset{(*)}{=} \sum_{n=0}^{\infty} (n+1)\cdot y^n,$$

valid for $|y|<1$

This series is obtained by differentiating the geometric series

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n, \text{ valid for } |y|<1$$

substitute $y = 4x$ in $(*)$

$$= \sum_{n=0}^{\infty} (n+1)\cdot 4^n \cdot x^{n+6}$$

valid for $|4x|<1$

i.e. $|x| < 1/4$.

8. $x = e^{\sin t}$, $y = \cos t + t - \pi$, $0 \le t \le 2\pi$

a) $P = (1,-1)$.

First, find corresponding value of parameter t:-

$x = e^{\sin t} = 1 \Rightarrow \sin t = 0 \Rightarrow t = 0, \pi$ or 2π

$y = \cos t + t - \pi = -1$: $\cos 0 + 0 - \pi = 1 - \pi$ ✗

$\cos \pi + \pi - \pi = -1$ ✓ so $t = \pi$.

$\cos 2\pi + 2\pi - \pi = 1 + \pi$ ✗

Slope of tangent line $m = \left.\frac{dy}{dx}\right|_{t=\pi} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi}$

$$= \left.\frac{(-\sin t + 1 - 0)}{\cos t \cdot e^{\sin t}}\right|_{t=\pi} = \frac{1}{-1\cdot e^0} = -1.$$

C.R.

Eq. of tangent line: $(y-(-1)) = (-1)\cdot(x-1)$, $\boxed{y = -x}$.

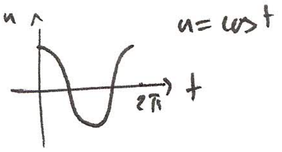
(eq. of line thru (a,b) w/ slope m is $(y-b) = m\cdot(x-a)$).

b. Vertical tangent line: $\frac{dx}{dt} = 0.$

$\cos t \cdot e^{\sin t} = 0$

$\cos t = 0$ ($e^{\sin t} \neq 0$ because $e^u \neq 0$ for all u)

$t = \pi/2, 3\pi/2$ ($0 \leq t \leq 2\pi$.

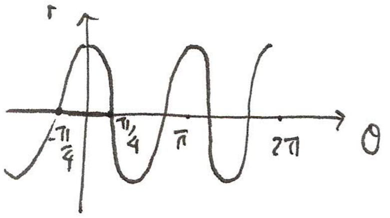


)

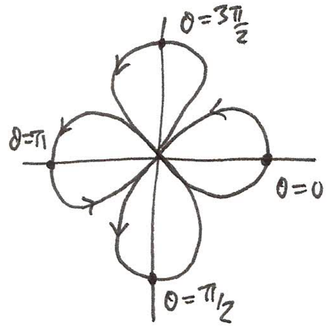
$\rightsquigarrow (x,y) = (e^{\sin t}, \cos t + t - \pi) = \Big((e^{1}, 0 + \pi/2 - \pi) = (e, -\pi/2)$

$\Big) (e^{-1}, 0 + 3\pi/2 - \pi) = (e^{-1}, \pi/2).$

9a. $r = \cos 2\theta$



$\rightsquigarrow$



The right hand loop corresponds to $-\pi/4 \leq \theta \leq \pi/4$.

So, area of RH loop $= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta$

$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (\cos 2\theta)^2 \, d\theta$

$\cos(2t) = 2(\cos t)^2 - 1$

$\Rightarrow (\cos t)^2 = \frac{1 + \cos(2t)}{2}$

$\Rightarrow (\cos 2\theta)^2 = \frac{1 + \cos(4\theta)}{2}$

$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{1 + \cos(4\theta)}{2} \, d\theta$

$= \frac{1}{4} \int_{-\pi/4}^{\pi/4} 1 + \cos(4\theta) \, d\theta$

$= \frac{1}{4} \Big[\theta + \frac{1}{4} \sin 4\theta \Big]_{-\pi/4}^{\pi/4}$

$= \frac{1}{4}\Big((\pi/4 + 0) - (-\pi/4 + 0) \Big) = \frac{1}{4} \cdot \frac{\pi}{2} = \pi/8.$

$\therefore$ area of all loops $= 4 \cdot \pi/8 = \boxed{\pi/2}$.

(Note that each loop has the same area because of the symmetry

$\theta \rightsquigarrow \theta + \pi/2$, $r \rightsquigarrow -r$: $\cos(2(\theta + \pi/2)) = \cos 2\theta \cdot \cos \pi - \sin\theta \cdot \sin \pi = -\cos 2\theta$.)

b. Slope of tangent line to curve $r = \cos 2\theta$ at $\theta = \pi/4$:–

$x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \underset{r=\cos 2\theta}{=} \frac{-2\sin 2\theta \cdot \sin\theta + \cos 2\theta \cos\theta}{-2\sin 2\theta \cdot \cos\theta - \cos 2\theta \sin\theta}$$

Slope m of tangent line at $\theta = \pi/4$:

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-2 \cdot 1 \cdot 1/\sqrt{2} + 0 \cdot 1/\sqrt{2}}{-2 \cdot 1 \cdot 1/\sqrt{2} - 0 \cdot 1/\sqrt{2}} = 1. \quad \square.$$

c.

$$L = \int_0^2 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta \underset{r=e^{3\theta}}{=} \int_0^2 \sqrt{e^{6\theta} + (3e^{3\theta})^2}\, d\theta$$

$$= \int_0^2 \sqrt{10 \cdot e^{6\theta}}\, d\theta = \sqrt{10} \cdot \int_0^2 e^{3\theta}\, d\theta$$

$$= \sqrt{10} \cdot \left[\tfrac{1}{3} e^{3\theta} \right]_0^2 = \frac{\sqrt{10}}{3} \cdot \left(e^6 - e^0 \right) = \frac{\sqrt{10}}{3} (e^6 - 1).$$