1. For $r > 0$,

$$\int_r^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_r^t 1/x^2\,dx = \lim_{t\to\infty} \left[-1/x\right]_r^t$$

$$= \lim_{t\to\infty} -1/t - (-1/r) = 1/r \quad \text{convergent.}$$

(Improper integral of type 1)

For $r \leq 0$ we also have a discontinuity of the integrand $1/x^2$ at $x = 0$ in the interval of integration. (Improper integral of type 2).

We have $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0_+} \int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0_+} \left[-1/x\right]_t^1$$

$$= \lim_{t\to 0_+} -1 - (-1/t) = \lim_{t\to 0_+} \frac{1}{t} - 1 = \infty.$$

So the integral is divergent for $r \leq 0$.

$\rightsquigarrow$ $\boxed{B}$

(More generally, $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent for $p > 1$
& divergent for $p \leq 1$

whereas $\int_0^1 \frac{1}{x^p}\,dx$ is convergent for $p < 1$
& divergent for $p \geq 1$.)

2. $$0.454545\ldots = 45\cdot\frac{1}{10^2} + 45\cdot\frac{1}{10^4} + 45\cdot\frac{1}{10^6} + \cdots = \sum_{n=1}^\infty \frac{45}{100}\cdot\left(\frac{1}{100}\right)^{n-1}$$

$$= a + ar + ar^2 + \cdots$$

where $a = \frac{45}{100}$ and $r = \frac{1}{100}$

$\boxed{B}$.

3.

A is true (Theorem 6 in 11.2)

B is false. For example, $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

C is true (Divergence test – 7 in 11.2. This is equivalent to the statement A ("contrapositive".))

D is true by the definition of convergence of a sequence:
$\{a_n\}_{n=1}^{\infty}$ converges means $\lim_{n\to\infty} a_n$ exists (& is finite).

$\rightsquigarrow$ $\boxed{B}$

4. I. $\sum_{n=1}^{\infty} \frac{2n}{n^3} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent (p-series, $p = 2 > 1$)

II. $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ divergent (p-series, $p = \frac{1}{2} \leq 1$)

III. $\sum_{n=1}^{\infty} \frac{n+8}{14n+9}$. Note $\lim_{n\to\infty} \frac{n+8}{14n+9} \underset{\substack{\text{divide top}\\ \text{\& bottom by } n}}{=} \lim_{n\to\infty} \frac{1+8/n}{14+9/n} \underset{\substack{\text{limit}\\ \text{laws}}}{=} \frac{1+0}{14+0} = \frac{1}{14} \neq 0.$

So the series $\sum_{n=1}^{\infty} \frac{n+8}{14n+9}$ is divergent by the divergence test.

$\rightsquigarrow$ $\boxed{C}$ I only.

5. $\sum_{n=0}^{\infty} \frac{3^{n-1}}{5^n} = \frac{3^{-1}}{5^0} + \frac{3^0}{5^1} + \frac{3^1}{5^2} + \dots = \frac{1}{3} + \frac{1}{3}\cdot\left(\frac{3}{5}\right) + \frac{1}{3}\cdot\left(\frac{3}{5}\right)^2 + \dots$

$= a + ar + ar^2 + \dots = \frac{a}{1-r} = \frac{1/3}{1-3/5} = \frac{1/3}{2/5} = 5/6.$

where $a = \frac{1}{3}$, $r = 3/5$, (& $|r| = 3/5 < 1 \Rightarrow$ convergent)

$\rightsquigarrow$ $\boxed{C}$.

(Alternatively, $\sum_{n=0}^{\infty} \frac{3^{n-1}}{5^n} = \sum_{n=0}^{\infty} 3^{-1} \cdot \left(\frac{3}{5}\right)^n = \frac{1}{3} + \frac{1}{3} \cdot \left(\frac{3}{5}\right) + \frac{1}{3} \cdot \left(\frac{3}{5}\right)^2 + \cdots$).

6. a. $\displaystyle\int \frac{x+1}{2x^2+x-3}\, dx$.

This integral is computed using the technique of partial fractions. (7.4)

(Note: 7.4 is <u>not</u> on the list of sections covered by our Exam 2)

$$\frac{x+1}{2x^2+x-3} = \frac{x+1}{(2x+3)(x-1)} = \frac{A}{2x+3} + \frac{B}{x-1}$$

$$x+1 = A \cdot (x-1) + B(2x+3)$$

$$x+1 = (A+2B)x + (-A+3B)$$

. Since this equation holds for all values of x, the coefficient of x & the constant terms on each side are equal.

$\rightsquigarrow$ ① $1 = A+2B$

② $1 = -A+3B$

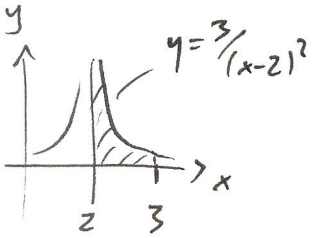
①+② $2 = 5B$, $B = 2/5$

① $\rightsquigarrow$ $A = 1-2B = 1-4/5 = 1/5$.

$$\int \frac{x+1}{2x^2+x-3}\, dx = \int \frac{1/5}{2x+3} + \frac{2/5}{x-1}\, dx = 1/5 \cdot \frac{1}{2} \ln|2x+3| + \frac{2}{5} \cdot \ln|x-1| + C.$$

$$= \frac{1}{10} \ln|2x+3| + \frac{2}{5} \ln|x-1| + C. \quad \square.$$

6b. $$\int_2^3 \frac{3}{(x-2)^2}\,dx = \lim_{t\to 2+} \int_t^3 \frac{3}{(x-2)^2}\,dx = \lim_{t\to 2+} \left[\frac{-3}{x-2}\right]_t^3$$



improper integral of type 2 : $3/(x-2)^2$ has infinite discontinuity at $x=2$.

$$= \lim_{t\to 2+} -3 - \left(\frac{-3}{t-2}\right)$$

$$= \lim_{t\to 2+} \frac{3}{t-2} - 3 = \infty.$$

So the integral is <u>divergent</u>.

7a. $$\sum_{n=2}^{\infty} \frac{(\sin n)^2}{n^2+1}$$

$$0 \leq \frac{(\sin n)^2}{n^2+1} \leq \frac{1}{n^2}$$ using $|\sin x| \leq 1$ for all x $\Rightarrow (\sin x)^2 \leq 1$.

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2>1$).

So $\sum_{n=2}^{\infty} \frac{(\sin n)^2}{n^2+1}$ converges by the Comparison Test.

b. $$\sum_{n=2}^{\infty} \frac{(n+1)^n}{2^{n+1}(-\ln n)^n} = \frac{1}{2}\sum_{n=2}^{\infty} \underbrace{\left(\frac{(n+1)}{2\cdot(-\ln n)}\right)^n}_{a_n}$$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n+1}{2\cdot \ln n} = \lim_{x\to\infty} \frac{x+1}{2\ln x} = \lim_{x\to\infty} \frac{1}{2/x}$$

l' Hôpital's rule

$$= \lim_{x\to\infty} \frac{x}{2} = \infty.$$

So $\sum_{n=2}^{\infty} \frac{(n+1)^n}{2^{n+1}(-\ln n)^n}$ is divergent by the root test.

8a. $\sum_{n=1}^{\infty} 6n^2 e^{-n^3}$

$$\int_1^\infty 6x^2 e^{-x^3}\,dx = 2\cdot\int_1^\infty e^{-x^3}\cdot 3x^2\,dx = 2\cdot\lim_{t\to\infty}\int_1^t e^{-x^3}\cdot 3x^2\cdot dx$$

$u = x^3$, $du = 3x^2\,dx$

$$= 2\cdot\lim_{t\to\infty}\int_1^{t^3} e^{-u}\,du$$

$$= 2\cdot\lim_{t\to\infty}\Big[-e^{-u}\Big]_1^{t^3} = 2\cdot\lim_{t\to\infty} -e^{-t^3} - (-e^{-1})$$

$$= 2\cdot\lim_{t\to\infty} e^{-1} - e^{-t^3} = 2\cdot e^{-1} = 2/e.$$

as $t\to\infty$, $-t^3 \to -\infty$,
so $e^{-t^3} \to 0$.

So $\int_1^\infty 6x^2 e^{-x^3}\,dx$ is convergent.

Also $f(x) = 6x^2 e^{-x^3}$ is positive, decreasing, and continuous

$$f'(x) = 12x\cdot e^{-x^3} + 6x^2\cdot(-3x^2)e^{-x^3} = (12x - 18x^4)e^{-x^3}$$

$$= 6x\cdot(2 - 3x^3)e^{-x^3} \leq 0 \quad \text{for } 2 - 3x^3 \leq 0$$

i.e. $x \geq \sqrt[3]{2/3}$.

So $f(x)$ is decreasing for $x \geq \sqrt[3]{2/3}$.

So $\sum_{n=1}^{\infty} 6n^2\cdot e^{-n^3}$ is convergent by the integral test.

(Alternative solution. $\sum_{n=1}^{\infty} 6n^2 e^{-n^3} = \sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot e^{-(n+1)^3+n^3}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \cdot \lim_{n\to\infty} e^{-3n^2-3n-1}$$

$$= 1 \cdot 0 = 0. < 1.$$

(note: as $n\to\infty$, $-3n^2-3n-1 \to -\infty$,
so $e^{-3n^2-3n-1} \to 0$.

So $\sum_{n=1}^{\infty} 6n^2 e^{-n^3}$ is convergent by the ratio test.).

8b. $$\sum_{n=2}^{\infty} \frac{\sqrt{n^2-2n+3}}{n^3+n+1}.$$

$$\frac{\sqrt{n^2-2n+3}}{n^3+n+1} \approx \frac{\sqrt{n^2}}{n^3} = \frac{n}{n^3} = \frac{1}{n^2} \text{ for large } n.$$

$$\lim_{n\to\infty} \frac{\sqrt{n^2-2n+3}}{n^3+n+1} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2 \cdot \sqrt{n^2-2n+3}}{n^3+n+1} \underset{\uparrow}{=} \lim_{n\to\infty} \frac{1 \cdot \frac{\sqrt{n^2-2n+3}}{n}}{1+\frac{1}{n^2}+\frac{1}{n^3}}$$

divide top & bottom by n^3

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{n^2-2n+3}{n^2}}}{1+\frac{1}{n^2}+\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\sqrt{1-\frac{2}{n}+\frac{3}{n^2}}}{1+\frac{1}{n^2}+\frac{1}{n^3}} \underset{\uparrow}{=} \frac{\sqrt{1-0+0}}{1+0+0} = 1, \text{ finite } \& \neq 0.$$

limit laws, $\sqrt{x}$ continuous

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent (p-series, $p=2$)

So $\sum_{n=2}^{\infty} \frac{\sqrt{n^2-2n+3}}{n^3+n+1}$ is convergent by the limit comparison test.

(note: $\frac{\sqrt{n^2-2n+3}}{n^3+n+1} \geq 0$ for all $n \geq 2$.)

9a. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+3}5^n}{3^n\cdot(2n)!}}_{a_n.}$ (Because of the factorial in a_n we use the ratio test.)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{5^{n+1}}{3^{n+1}\cdot(2n+2)!}\cdot\frac{3^n\cdot(2n)!}{5^n}\right) \overset{*}{=} \lim_{n\to\infty}\frac{5}{3\cdot(2n+2)\cdot(2n+1)} = 0<1$$

So the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+3}\cdot 5^n}{3^n\cdot(2n)!}$ converges absolutely by the ratio test

[* More details: Note $\dfrac{(2n)!}{(2n+2)!} = \dfrac{2n\cdot(2n-1)\cdots 3\cdot2\cdot1}{(2n+2)\cdot(2n+1)\cdot 2n\cdots 3\cdot2\cdot1} = \dfrac{1}{(2n+2)(2n+1)}.$]

9b. $\displaystyle\sum_{n=1}^{\infty}\underbrace{(-1)^n\cdot\frac{n+1}{n^2+7}}_{a_n}$

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n+1}{n^2+7}.$$

$\dfrac{n+1}{n^2+7} \approx \dfrac{n}{n^2} = \dfrac{1}{n}$ for large n.

$$\lim_{n\to\infty}\frac{n+1}{n^2+7}\Big/\frac{1}{n} = \lim_{n\to\infty}\frac{n^2+n}{n^2+7} \underset{\substack{\text{divide top \&}\\ \text{bottom by } n^2}}{=} \lim_{n\to\infty}\frac{1+1/n}{1+7/n^2} \underset{\substack{\text{limit}\\ \text{laws}}}{=} 1$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. So $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2+7}$ is divergent by the limit comparison test.

So $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n+1}{n^2+7}$ is NOT absolutely convergent.

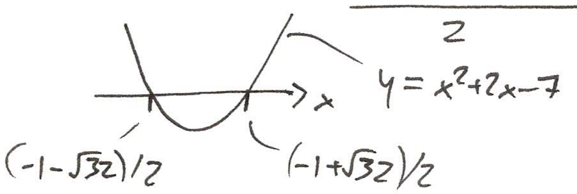
$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n+1}{n^2+7} = -\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \underbrace{\frac{n+1}{n^2+7}}_{b_n}$$

$b_n \geq 0$, b_n decreasing & $b_n \to 0$ as $n \to \infty$: $\lim_{n\to\infty} \frac{n+1}{n^2+7} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{1+7/n^2} = \frac{0+0}{1+0} = 0.$

$f(x) = \frac{x+1}{x^2+7}$ $\quad f'(x) = \frac{1\cdot(x^2+7) - (x+1)\cdot 2x}{(x^2+7)^2} = \frac{7-2x-x^2}{(x^2+7)^2} \leq 0$

for $7-2x-x^2 \leq 0$, i.e. $x^2+2x-7 \geq 0$. This holds for x sufficiently large

(More precisely, for $x \geq \frac{-1 \pm \sqrt{2^2 - 4\cdot(-7)}}{2} = \frac{-1+\sqrt{32}}{2}$ by the quadratic formula.)



So $f(x)$ is increasing for $x \geq \frac{-1+\sqrt{32}}{2}$.

Alternatively: check $b_n \geq b_{n+1}$ directly:

$$b_n - b_{n+1} = \frac{n+1}{n^2+7} - \frac{(n+1)+1}{(n+1)^2+7} = \frac{(n+1)\cdot(n^2+2n+8) - (n+2)(n^2+7)}{(n^2+7)(n^2+2n+8)}$$

$$= \frac{(n^3+3n^2+10n+8) - (n^3+2n^2+7n+14)}{(n^2+7)(n^2+2n+8)} = \frac{n^2+3n-6}{(n^2+7)(n^2+2n+8)} \geq 0$$

for $n \geq \frac{-3+\sqrt{9+24}}{2}$

using quadratic formula for numerator.

So, by the alternating series test,

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n+1}{n^2+7} \text{ is convergent.}$$

Combining our results, $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n+1}{n^2+7}$ is conditionally convergent

(convergent, but not absolutely convergent.) $\square$.