1.
$$a_n = \frac{n^3}{3n^3+5}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{3n^3+5} \underset{\uparrow}{=} \lim_{n\to\infty} \frac{1}{3+5/n^3} = \frac{1}{3}$$

divide top & bottom by n^3

So the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

But the series $\sum_{n=1}^{\infty} a_n$ diverges by the divergence test

(because $\lim_{n\to\infty} a_n \neq 0$).

$\boxed{b}$.

2. The divergence test states: If $\lim_{n\to\infty} a_n \neq 0$ or does not exist

then $\sum_{n=1}^{\infty} a_n$ diverges.

So I & II are true.

III is false: for example, if $a_n = \frac{1}{n}$ then $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} a_n$ diverges.

$\boxed{a}$

3. The comparison test states: If $0 \leq a_n \leq b_n$ then

1. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges

2. If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

The limit comparison test states: If ~~$0 \leq a_n \leq b_n$~~ $0 \leq a_n$, $0 \leq b_n$ and

$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$ then $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$

both converge or both diverge.

So c is true (#1 of comparison test).

a is not necessarily true. e.g. $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$. $0 \leq a_n \leq b_n$.

$\sum a_n$ converges, $\sum b_n$ diverges.

b is not necessarily true. e.g. $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$ as above.

d is not necessarily true e.g. $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$ again.

(note LCT requires $c \neq 0$!)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

$\boxed{c}$.

4. $$\sum_{n=1}^{\infty} (-1)^n \cdot 7 \cdot \frac{4^n}{3^n} = \sum_{n=1}^{\infty} 7 \cdot \left(-\frac{4}{3}\right)^n$$

geometric series, common ratio $r = -4/3$

$|r| = 4/3 \geq 1$

So the series is <u>divergent</u>.

$\boxed{d}$.

5. The root test states:

If $\lim_{n\to\infty} |a_n|^{1/n} = L$, then

- $\sum_{n=1}^{\infty} a_n$ is absolutely convergent for $L < 1$
- $\sum_{n=1}^{\infty} a_n$ is divergent for $L > 1$ OR $L = \infty$
- the test is inconclusive for $L = 1$

$\boxed{a}$.

6. a) $\displaystyle\int \frac{y}{y^2-2y-3}\,dy$

Use 7.4 Partial fractions (not covered by our Exam 2)

$$\frac{y}{y^2-2y-3} = \frac{y}{(y-3)(y+1)} = \frac{A}{y-3} + \frac{B}{y+1}$$

$$y = A(y+1) + B(y-3)$$

$$1\cdot y + 0 = y = (A+B)y + (+A-3B)$$

① $1 = A+B$ ① − ②: $1 = 4B$, $B = 1/4$

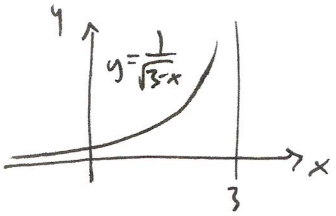
② $0 = A-3B$ ①: $A = 1-B = 3/4$.

So $$\int \frac{y}{y^2-2y+3}\,dy = \int \frac{3/4}{y-3} + \frac{1/4}{y+1}\,dy$$

$$= \tfrac{3}{4}\ln|y-3| + \tfrac{1}{4}\ln|y+1| + C.$$

b). $$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3_-} \int_2^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3_-} \int_1^{3-t} \frac{1}{\sqrt{u}}\cdot -du$$

$u = 3-x$ $dx = -du$



$\frac{1}{\sqrt{3-x}}$ has infinite discontinuity at $x=3$ ⇝ improper integral of type 2.

$$= \lim_{s\to 0_+} \int_s^1 \frac{1}{\sqrt{u}}\,du = \lim_{s\to 0_+} \left[\frac{u^{1/2}}{1/2}\right]_s^1 = \lim_{s\to 0_+} 2.(1-\sqrt{s})$$

$$= 2.$$ <u>convergent.</u>

here I write $s = 3-t$

& switched the limits

(recall $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$)

7a. $$\sum_{n=0}^{\infty} 3^{n+1} \cdot (x+4)^n = \sum_{n=0}^{\infty} 3 \cdot (3 \cdot (x+4))^n$$

geometric series, common ratio $r = 3 \cdot (x+4)$.

Convergent if and only if $|r| < 1$

$$|3(x+4)| < 1$$

$$-1 < 3(x+4) < 1$$

$$-1/3 < x+4 < 1/3$$

$$\boxed{-4\tfrac{1}{3} < x < -3\tfrac{2}{3}}$$

b. $$\sum_{n=1}^{\infty} \frac{(\cos n)^2}{5^n}$$

$$0 \leq \frac{(\cos n)^2}{5^n} \leq \frac{1}{5^n}$$ because $|\cos x| \leq 1$ for all x $\Rightarrow (\cos x)^2 \leq 1$.

$\sum_{n=1}^{\infty} \frac{1}{5^n}$ is convergent : geometric series, common ratio $r = 1/5$, $|r| = 1/5 < 1$.

So $\sum_{n=1}^{\infty} \frac{(\cos n)^2}{5^n}$ is convergent by the comparison test.

8a. $$\sum_{n=1}^{\infty} \frac{4n}{6n^2+2n+8}$$ $\frac{4n}{6n^2+2n+8} \geq 0$ ✓

$$\frac{4n}{6n^2+2n+8} \approx \frac{4n}{6n^2} = \frac{2}{3} \cdot \frac{1}{n}$$ for n large.

$$\lim_{n\to\infty} \frac{4n}{6n^2+2n+8} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{4n^2}{6n^2+2n+8} \underset{\substack{\text{divide top \&} \\ \text{bottom by } n^2}}{=} \lim_{n\to\infty} \frac{4}{6+\frac{2}{n}+\frac{8}{n^2}} \underset{\substack{\text{limit} \\ \text{laws}}}{=} \frac{4}{6+0+0} = \frac{2}{3} \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p=1 \leq 1$)

So $\sum_{n=1}^{\infty} \frac{4n}{6n^2+7n+8}$ diverges by the limit comparison test.

8b. $\sum_{n=3}^{\infty} \frac{\ln(n^2)}{n}$

$\frac{\ln(n^2)}{n} \geq \frac{1}{n} \geq 0$ for all $n \geq 2$ $(\ln(n^2) \geq 1$ for $n^2 \geq e = 2.718..)$

$\sum_{n=3}^{\infty} \frac{1}{n}$ is divergent (p-series, $p=1 \leq 1$).

So $\sum_{n=3}^{\infty} \frac{\ln(n^2)}{n}$ is divergent by the comparison test.

9.a. $\sum_{n=1}^{\infty} \frac{n^3 \cdot (n+1)!}{e^n \cdot n!} = \sum_{n=1}^{\infty} \underbrace{\frac{n^3 \cdot (n+1)}{e^n}}_{a_n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3 \cdot (n+2)}{e^{n+1}} \cdot \frac{e^n}{n^3 \cdot (n+1)} = \lim_{n\to\infty} \frac{(n+1)^2 (n+2)}{n^3} \cdot \frac{1}{e}$

$= \lim_{n\to\infty} (1+1/n)^2 \cdot (1+2/n) \cdot \frac{1}{e} = \frac{1}{e} < 1.$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^3 (n+1)!}{e^n \cdot n!}$ is absolutely convergent by the ratio test.

9b. $\sum_{n=2}^{\infty} \underbrace{(-1)^{n+1} \cdot \frac{5}{3\sqrt{n}-3}}_{a_n}$

$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{5}{3\sqrt{n}-3}$ $\qquad \frac{5}{3\sqrt{n}-3} \geq \frac{5}{3\sqrt{n}} = \frac{5}{3} \cdot \frac{1}{n^{1/2}}$

$\sum_{n=2}^{\infty} 1/n^{1/2}$ is divergent (p-series, $p = 1/2 \leq 1$).

So $\sum_{n=2}^{\infty} \frac{5}{3\sqrt{n}-3}$ is divergent by comparison test.

So $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{5}{3\sqrt{n}-3}$ is NOT absolutely convergent.

$$\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \underbrace{\frac{5}{3\sqrt{n}-3}}_{b_n} = -b_2 + b_3 - b_4 + \cdots = -(b_2 - b_3 + b_4 - \cdots)$$

alternating series.

$b_n \geq 0$, b_n decreasing (because $\sqrt{n}$ increasing), $\lim_{n\to\infty} b_n = 0$ (because $\lim_{n\to\infty} \sqrt{n} = \infty$)

So, by alternating series test, $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{5}{3\sqrt{n}-3}$ is convergent.

Combining, $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{5}{3\sqrt{n}-3}$ is <u>conditionally convergent</u>

(convergent, but not absolutely convergent). $\square$.